# Pure injective modules over a commutative valuation domain

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**Abstract**

Using geometrical invariants we classify those pure injective modules over a commutative valuation domain which are envelopes of one element.

## 1 Introduction

The problem of classification of pure injective modules over a commutative valuation domain (CVD) was posed in the book of Fuchs and Salce [2, Probl. 11]. A complete description of indecomposable pure injective modules over a CVD is due to Ziegler [11]. So the main difficulty in the above mentioned classification is provided by the so-called superdecomposable pure injective modules, where a module $M$ is called *superdecomposable* if $M$ does not contain an indecomposable direct summand.

Superdecomposable pure injective modules over a commutative valuation domain (CVD) were first mentioned in [2, Ch. 11]. But, as was noticed later, even the existence of these had not been proved there. The first complete proofs appeared in Puninski [5] and Salce [9]. In particular from [5] it follows that over a CVD $V$ a superdecomposable pure injective module exists iff $V$ does not have Krull dimension (in the sense of Gabriel and Rentschler) and a similar criterion was found in [9].

By Prest [8, Ch. 4], every element $m$ of a pure injective module $M$ over any ring is contained in a (unique) "minimal" direct summand $N(m)$ of $M$. Also every pure injective module is a pure injective envelope of a module

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$\oplus_{i\in I} N(m_i)$. So the classification of pure injective modules of the form $N(m)$ is an essential ingredient in a solution of the general classification problem.

In this paper, using geometrical invariants, we classify pure injective modules $N(m)$ over a CVD $V$. It is known (see indicator functions in [2, Ch. 11] or [5]) that to every element $m$ in a pure injective module $M$ over a CVD $V$ one can assign a function $f : \Gamma \to \widehat{\Gamma}$ ($\Gamma$ is a positive cone of the valuation group of $V$ and $\widehat{\Gamma}$ is its completion by cuts) such that $N(m)$ is completely determined by $f$. Of course different functions could lead to the same pure injective module.

We describe (geometrically) an equivalence relation $\sim$ on functions such that $f \sim g$ iff the corresponding modules $N(f)$ and $N(g)$ are isomorphic. From this follows an easy description of a decomposition $N(f) = N(g) \oplus N(h)$ in terms of these functions which yields that $N(g)$ and $N(h)$ do not have any direct summand in common (model theorists say "orthogonal" in this case).

As a direct consequence of this result we prove that for every module $N(m)$, its endomorphism ring $S = \mathrm{End}(N(m))$ is abelian regular after factorization by its Jacobson radical $\mathrm{Jac}(S)$. By Zimmermann-Huisgen and Zimmermann [12, Thm. 9] for every pure injective module $M$ over any ring with $S = \mathrm{End}(M)$, $S/\mathrm{Jac}(S)$ is a von Neumann regular right self-injective ring and idempotents can be lifted modulo $\mathrm{Jac}(S)$. We show that for every pure injective module $M$ over a CVD, $S/\mathrm{Jac}(S)$ is of type I in the terminology of Goodearl [4, Ch. 10]. Also ${}_SM$ is a Bezout module, i.e. the sum of two cyclic $S$-submodules of $M$ is a cyclic module. Note that an essential part of the arguments in [6] was to prove that for every pure injective module $M$ over a CVD, ${}_SM$ is a distributive module.

The above result gives also the possibility for a direct application of the well developed theory of nonsingular injective modules over a von Neumann regular ring. Unfortunately we are not able to describe clearly a connection between the list of invariants given by Goodearl's theory and our geometrical description. So this is the task for future.

## 2 Preliminaries

A *commutative valuation domain* (CVD) $V$ is a commutative domain whose ideals are linearly ordered by inclusion. This means that for every $a, b \in V$, either $a \in bV$ or $b \in aV$ holds. For instance $\mathbb{Z}_{(p)}$ (the localization of the integers $\mathbb{Z}$ at a prime ideal $p\mathbb{Z}$) is a CVD. For elements $a, b$ of a CVD $V$

we put $a \leq b$ if $bR \subseteq aR$ and $a < b$ if $bR \subset aR$. This order corresponds to a natural order on integers: say, for $p, p^2 \in \mathbb{Z}_{(p)}$, we have $p < p^2$.

Since every CVD $V$ is a local ring, the set of noninvertible elements of $V$ coincides with its Jacobson radical $\mathrm{Jac}(V)$, and $R \setminus \mathrm{Jac}(R) = \mathrm{U}(R)$ is the set of units of $V$. For $a, b \in V$ we write $a \sim b$ if $aR = bR$ which is clearly the same as $a \leq b \leq a$ or $a = bu$, $u \in \mathrm{U}(V)$. Factorizing $V$ by $\sim$ we obtain an ordered abelian semigroup $(\Gamma(V), \leq)$ with cancellation and a natural map (evaluation) $a \to v(a) \in \Gamma(V)$ such that $v(ab) = v(a) + v(b)$ holds for $0 \neq a, b \in V$. The largest element of $\Gamma(V)$ is given by $v(0)$ (and one often writes $\infty$ instead) and the smallest element of $\Gamma(V)$ is $v(1)$. In fact $\Gamma(V)$ can be converted into the ordered abelian group by the usual procedure, but we do not need this fact in the paper. For instance for $V = \mathbb{Z}_{(p)}$, $\Gamma(V)$ is (ordered) isomorphic to $(\mathbb{N}, +, \leq)$.

A *cut* on $\Gamma(V)$ is an arbitrary partition $\Gamma(V) = A \cup B$ such that $A \neq \emptyset$, and $B$ is a filter on $\Gamma(V)$, i.e. $b \in B$ and $b \leq c$ implies $c \in B$. Then $A$ is clearly an ideal of $\Gamma(V)$, i.e. $a \in A$ and $b \leq a$ yields $b \in A$. The set $\widehat{\Gamma}$ of cuts on $\Gamma$ can be linearly ordered by the rule $(A, B) \leq (A', B')$ if $A \subseteq A'$ (equivalently $B' \subseteq B$). There is a natural embedding $\Gamma \to \widehat{\Gamma}$ where $a \in \Gamma$ goes to the cut $\hat{l} = \hat{l}(a) = \{b \in \Gamma \mid b \leq a\}$ and $B(\hat{l}) = \Gamma \setminus A(\hat{l})$. For instance $\hat{l}(0)$ is the largest cut $\infty$ with $A(\infty) = \Gamma$.

For background in the model theory of modules the reader is referred to M. Prest's book [8]. For instance a module $M$ is *pure injective* if it is injective with respect to pure embeddings. As with injective envelopes, every module $M$ has a unique *pure injective envelope* $\mathrm{PE}(M)$. A *pp-formula* is an existentially quantified formula $\varphi(x_1, \ldots, x_m)$ "there exists $\bar{y} = (y_1, \ldots, y_k)$ such that $\bar{y}A = \bar{x}B$", where $A$ is a $k \times l$ and $B$ is a $m \times l$ matrix over a ring. We say that $\varphi$ is satisfied by a tuple $\overline{m} \in M$, written $M \models \varphi(\overline{m})$, if there exists a tuple $\bar{n} \in M$ such that $\bar{n}A = \overline{m}B$. Every pp-formula $\varphi(x)$ defines on $M$ a *pp-definable subgroup* $\varphi(M) = \{m \in M \mid M \models \varphi(m)\}$ and $\varphi(M)$ is even an $S = \mathrm{End}(M)$-submodule of $M$ (hence a submodule of $M$ if the ring is commutative). If $\varphi, \psi$ are pp-formulae we say that $\varphi$ implies $\psi$, written $\varphi \to \psi$, if for any module $M$, $\varphi(M) \subseteq \psi(M)$.

Given $m \in M$, a *pp-type* $pp_M(m)$ is a collection of pp-formulae $\{\varphi \mid M \models \varphi(m)\}$. A pp-type can be also described as a set of pp-formulae that is closed via finite conjuctions and implications. Given a pp-type $p$, there is a unique "minimal" pure injective module $(N(p), m)$ such that $pp_{N(p)}(m) = p$. For instance $N(p)$ is a direct summand of every pure injective module realizing $p$.

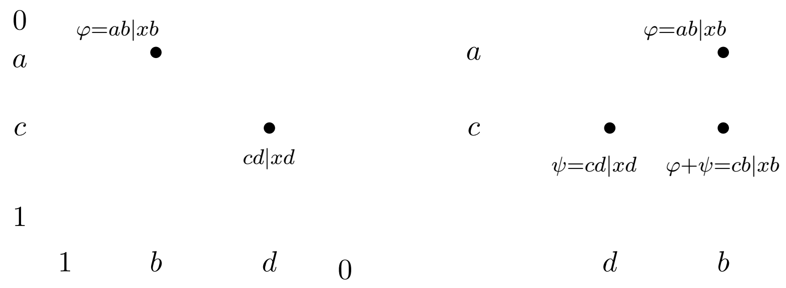


Figure 1:

## 3 Geometrical description of pp-types

Since we will use only pp-formulae of a very special kind, let us describe them more explicitly. A *divisibility* formula is a formula $a \mid x$ (there exist $y$ such that $ya = x$), hence it defines the submodule $Ma$ in any $V$-module $M$. Clearly for $a \leq b \in V$ we have $b \mid x \to a \mid x$ and the converse is also true. An *annihilator formula* is a pp-formula $xb = 0$, $b \in V$ and it defines in every module $M$ the submodule $(xb = 0)(M) = \{m \in M \mid mb = 0\}$. Similarly annihilator formulae over a CVD $V$ form a chain, where $xa = 0 \to xb = 0$ iff $a \leq b$.

It is not difficult to see that the sum of pp-formulae $a \mid x + xb = 0$ is a pp-formula $ab \mid xb$ which defines in a module $M$ the submodule $(ab \mid xb)(M) = \{m \in M \mid mb \in Mab\}$. To every pp-formula $ab \mid xb$ we assign a point $(b, a)$ of the plane $\Gamma \times \Gamma$, where the divisibility formula $a \mid x$ goes to the point $(1, a)$ on the $y$ axis and an annihilator pp-formula $xb = 0$ goes to the the point $(b, 1)$ on the $x$ axis. It follows from [5] that implication among pp-formulae $ab \mid xb$, $a, b \in V$ acts "right and down", i.e. the set of consequences of a pp-formula $\varphi$ is contained in the angle with $\varphi$ on the top (see Figure 1 on the left).

Also the above decomposition shows that the sum of two pp-formulae $ab \mid xb$ and $cd \mid xd$ can be drawn as in Figure 1 on the right.

The lattice of all pp-formulae over a CVD is generated by the two chains just described, hence it is distributive. Every 1-pp-formula over $V$ is equivalent to a finite conjuction of pp-formulae $\varphi_i = a_i b_i \mid x b_i$. Moreover every implication among them is "free" meaning that $\wedge_{i=1}^{n} \varphi_i \to \varphi = ab \mid xb$ iff $\varphi_i \to \varphi$ for some $i$. Also every pp-formula $\varphi(x_1, \ldots, x_n)$ over $V$ is equivalent

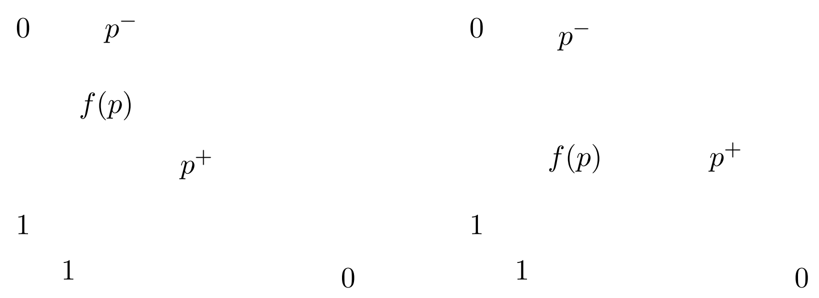


Figure 2:

to a finite conjunction of (divisibility) formulae $a \mid xb_1 + \ldots + xb_n$, $a, b_i \in V$.

It follows that every pp-type $p(x)$ over $V$ is uniquely determined by pp-formulae $ab \mid xb \in p$, i.e. by some subset of the plane $\Gamma \times \Gamma$. Let us make this description more precise. Let $p = p(x)$ be a pp-type (in one free variable) over a CVD $V$. We construct from $p$ a function $f(p) : \Gamma \to \widehat{\Gamma}$ by setting $A(f(b)) = \{a \in \Gamma : ab \mid xb \in p\}$. For instance $A(f(b)) = \infty$ iff $xb = 0 \in p$. Then (see [7, Ch. 12]) 1) $f$ is nondecreasing; 2) $f(1) \neq \infty$ and 3) $f(0) = \infty$. Moreover there is 1-1 correspondence between such functions and 1-pp-types over a CVD $V$. Thus we obtain a geometrical representation of every 1-pp-type $p$ as the graph of the function $f(p)$. Here (see Figure 2 on the left) the positive part $p^+$ of $p$ is under the graph of $f(p)$, and the negative part $p^-$ is over $f(p)$.

A pp-type $p$ is called *indecomposable* if the module $N(p)$ is indecomposable and $p$ is *superdecomposable* if $N(p)$ is a superdecomposable module. By [1, p. 162] in terms of this description $p$ is indecomposable iff $f(p)$ is a one step ladder (see Figure 2 on the right). On the level of pp-formulae that means that $ab \mid xb \in p$ implies either $a \mid x \in p$ or $xb = 0 \in p$.

The property of $p$ being superdecomposable can also be reformulated in purely geometrical terms (see [7, Ch. 12]). Precisely $p$ is superdecomposable iff for every pp-formula $\varphi = ab \mid xb \in p^-$, there is a rectangle with $\varphi$ at its left upper corner such that only the lower right corner of it is in $p^+$ (see Figure 3 on the left).

If a superdecomposable pure injective module exists, $\Gamma$ must contain a copy of the ordered set of rationals $(\mathbb{Q}, \leq)$. For instance if $\Gamma$ is a dense linear order (for every $a < b \in \Gamma$, $a < c < b$ for some $c$), then the diagonal $y = x$ (i.e. $ab \mid xb \in p$ iff $a \leq b$) is such.


Figure 3:

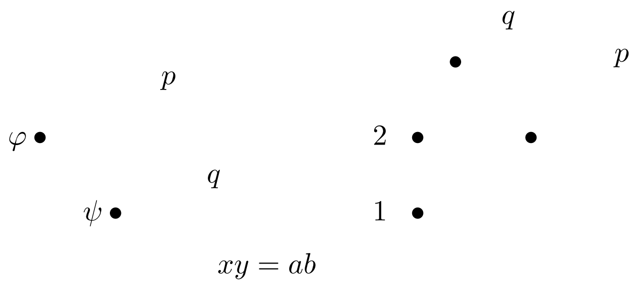


Figure 4:

## 4 Isomorphism criterion for $N(p)$

Let $p(x)$ be a pp-type over a CVD $V$. In this section we describe the pp-types $q$ that are realized in $N(p)$.

Let $q, r$ be pp-types over $V$. We say that $q$ and $r$ are *equivalent over a pp-formula* $\varphi$ if $\varphi \in q^-, r^-$ and for every pp-formula $\psi$ such that $\varphi \to \psi$, $\psi \in q$ iff $\psi \in r$ (so $q$ and $r$ look similar over $\varphi$). Geometrically that means that the $\varphi$-neighborhoods of $q$ and $r$ coincide (see Figure 3 on the right).

Let $p, q$ be pp-types over $V$, $\varphi = ab \mid xb$, $\psi = cd \mid xd$ such that $\varphi \in p^-$, $\psi \in q^-$. We say that $\psi$-neighborhood of $q$ is obtained by translation of the $\varphi$-neighborhood of $p$, if $ab = cd$ (i.e. $v(a) + v(b) = v(c) + v(d)$) and the former is obtained from the latter by a translation along the line $v(x) + v(y) = v(a) + v(b)$ (see Figure 4 on the left).

The following proposition is a criterion for: $N(q)$ is a direct summand


Figure 5:

of $N(p)$.

**Proposition 4.1** *Let $p, q$ be 1-pp-types over a commutative valuation domain $V$. Then $q$ is realized in $N(p)$ iff for every $\varphi' = a'b' \mid xb' \in q^-$ there exists $\varphi = ab \mid xb \in q^-$ and $\psi \in p^-$ such that $\varphi' \to \varphi$ and the $\psi$-neighborhood of $p$ is obtained from the $\varphi$-neighborhood of $q$ by a translation along the line $v(x) + v(y) = v(a) + v(b)$.*

In order to clarify this condition, let us consider an example. Let $p$ be the pp-type given by the line $y = x$ and let $q$ be given by the line $y = x+2$ as shown in Figure 4 on the right (we assume that $\Gamma$ looks like the non-negative rationals or reals). Then $q$ is realized in $p$, i.e. $N(q)$ is a direct summand of $N(p)$ but $N(p)$ is not a direct summand of $N(q)$, in particular these modules are nonisomorphic. Indeed, if $(b, a) \in q^-$ then clearly $(b + 1, a - 1) \in p^-$. On the other hand no neighborhood of $(0, 1) \in p^-$ can be translated to an isomorphic neighborhood of $q$.

Note that if $m \in N(p)$ is a realization for $p$ and $r \in V$ is such that $v(r) = 2$, then the pp-type of $mr$ in $N(p)$ is $q$. In particular there is a pure embedding $N(q) \to N(p)$ over $mr$ whose image is a direct summand of $N(p)$. Nevertheless (see below) under projection to this direct summand, the image of $m$ has pp-type not equal to $q$.

For pp-types $p$ and $q$ as shown in Figure 5 on the left, $N(p) \cong N(q)$ but we should first move $\varphi'$ and only then apply a translation.

**Proof.** Let us prove the necessity. To distinguish $p$ and $q$ we will assume that $p = p(x)$ and $q = q(y)$. Since $q$ is realized in $N(p)$, then (see [8, Ch. 6])

$q$ is maximal over $p$, i.e. there is a pp-type $r(y,x)$ which is consistent with $p(x) \cup q(y)$ (i.e. no formula of $p^-$ or $q^-$ is a consequence of $r \cup p(x) \cup q(y)$), and for every $\varphi(y) \in q^-$ there is $\psi(x) \in p^-$ such that $\varphi \cup r \to \psi$.

Let $\varphi' = a'b' \mid yb' \in q^-$. Then there are pp-formulae $\theta(y,x) \in r$ and $\psi(x) \in p^-$ such that $\varphi' \wedge \theta \to \psi$. We may assume that $\theta = \wedge_i a_i \mid xb_i + yc_i$, $a_i, b_i, c_i \in V$ and $\psi = a \mid xb$. By the common denominator theorem [7, Ch. 10], this implication can be decomposed as:

$$\varphi' = a'b' \mid yb' \to a'b'g \mid yb'g \to a \mid yb'g,$$

where $a'b'g = ua$,

$$\theta_i = a_i \mid yb_i + xc_i \to a_i g_i \mid yb_i g_i + xc_i g_i \to a \mid yb_i g_i + xc_i g_i,$$

where $a_i g_i = g_i' a$ (we have obtained a common denominator $a$),

$$\varphi' \wedge \theta \to a \mid y(\sum b_i g_i + b'g) + x(\sum c_i g_i)$$

and the last formula implies $a \mid xb$ in view of $\sum b_i g_i + b'g = sa$ and $\sum c_i g_i = b + ta$.

We set $\theta' = a \mid y(\sum b_i g_i - sa) + x(\sum c_i g_i - ta)$, i.e. $\theta' = a \mid yb'g + xb$. Then $\theta \to \theta'$ hence we may assume that $\theta = \theta'$. Also $\varphi' \to \varphi = a \mid yb'g$ and $\varphi \in q^-$ (otherwise $r$ is not consistent with $p \cup q$). Then all formulae $\varphi(y) = a \mid yb'g \in q^-$, $\theta'(x,y) = a \mid yb'g + xb \in r$ and $\psi(x) = a \mid xb \in p^-$ are on the same line $v(x) + v(y) = v(a)$.

Let us prove that the $\varphi$-neighborhood of $q$ and the $\psi$-neigborhood of $p$ are isomorphic via this line. Indeed if $\varphi'(y) \in q^-$ is in the neighborhood of $\varphi$, then the implication $\varphi \to \varphi'$ can be decomposed in two steps: right and then down (see Figure 5 on the right). In ring language this means that we multiply $a$ and $b'g$ by $t \in V$ and then we divide $at$ by an element of $V$ moving to the line $v(x) + v(y) = v(a')$. Repeating this for $\theta$ and $\psi$ we get formulae $\theta'(y,x)$ and $\psi'$ on the same line, where, since $\theta \to \theta'$, $\theta' \in r$. If $\psi' \in p$, then $\theta' \wedge \psi' \to \varphi'$ yields $\varphi' \in r$, i.e. $\varphi' \in q$, a contradiction. Arguing similarly for the pp-formula $\varphi' \in q^-$, we get the required isomorphism.

Let us prove sufficiency. A formula $\theta(y,x) = a \mid yb + xc$ will be called *connecting*, if $\varphi(y) = a \mid yb \in q^-$, $\psi(x) = a \mid xc \in p^-$ and the $\varphi$-neighborhood of $q$ and the $\psi$-neighborhood of $p$ are isomorphic along the line $v(x) + v(y) = v(a)$. The projections $\varphi(y)$, $\psi(x)$ will be also called *connecting* formulae. For connecting formulae $\varphi(y), \varphi'(y) \in q^-$ set $\varphi \sim \varphi'$ if $\varphi + \varphi' \in q^-$, hence the isomorphism of neighborhoods of $q$ and $p$ can be


Figure 6:

extended to a larger $\varphi''$-neighborhood of $q$ (see Figure 6 on the right). We will use the same symbol $\sim$ for the transitive closure of this relation.

Now we construct a pp-type $r(y,x)$ in the following way. Choose a representative $\theta(y,x)$ from every equivalence class of $\sim$ and multiply it by moving according to the definition of $\sim$. Now add $p(x) \cup q(y)$. It is almost evident that this type is consistent with $p \cup q$ and has the desired properties (there is no interference between formulas in different $\sim$-classes exept what is obvious, i.e. that given by $p \cup q$). □

Note that (see [3, Cor. 2]) pure injective modules $M$ and $N$ are isomorphic iff $M$ is a direct summand of $N$ and $N$ is a direct summand of $M$. Thus Proposition 4.1 answers the question of when modules $N(p)$ and $N(q)$ are isomorphic.

Nevertheless it is not easy to describe the shapes which a function could have in a given equivalence class. We say that a function $f$ is *rigid* if $N(f) \cong N(g)$ yields $f = g$. For instance the answer to the following question seems to depend on the existence of a kind of fractal structure.

**Question 4.2** *Let $V$ be a commutative valuation domain such that $\Gamma(V) \cong \mathbb{Q}^+$. Is it true that the function $y = x$ is rigid ?*

# 5 Decompositions of $N(p)$

Let $N(p)$ be a pure injective module over a CVD, where $m \in N(p)$ realizes $p$. Assume that $N(p) = N_1 \oplus N_2$ and that $m = m_1 + m_2$ via this decomposition. Then for $q = pp_{N(p)}(m_1)$, $r = pp_{N(p)}(m_2)$ by [8, Ch. 4] we have $N_1 = N(q)$,

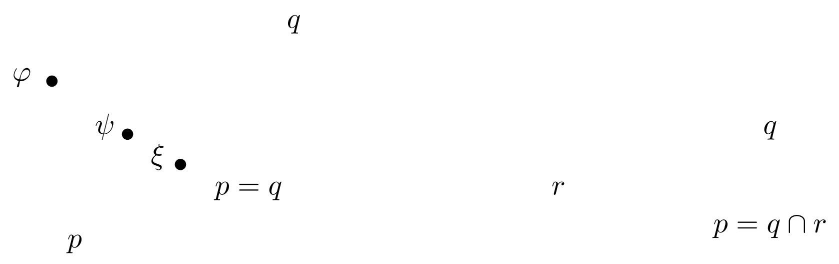


Figure 7:

$N_2 = N(r)$ and clearly $p = q \cap r$. We will refer to such a decomposition of $N(p)$ (and of $p$) as *canonical*.

Let us refine Proposition 4.1 for a canonical decomposition of $N(p)$.

**Lemma 5.1** *Let $p, q, r$ be pp-types over a CVD $V$ such that $N(p) = N(q) \oplus N(r)$ is a canonical decomposition. Then for every $\varphi = a' \mid xb' \in q^-$ there exists $\psi = a \mid xb \in q^- \cap r^+$ such that $\varphi \to \psi$ and $q \sim p$ over $\psi$ (see Figure 7 on the left).*

**Proof.** Let $m \in N = N(p)$ realize $p$ and let the decomposition $N(p) = N(q) \oplus N(r)$ induce a decomposition $m = n + k$. Thus $q(y) = pp(n)$, $r(z) = pp(k)$ and $p = q \cap r$.

Arguing as in the proof of Proposition 4.1, we find $\varphi' = a \mid yb \in q^-$ and $\theta(y, x) = a \mid yb + xc$ such that $\varphi \to \varphi'$, $N(p) \models \theta(n, m)$ and $\psi(x) = a \mid xc \in p^-$. Projecting $\theta(n, m)$ onto $N(r)$ we get $\psi(z) \in r^+$. Since $\psi(x) \in p^-$, we have $\psi(y) \in q^-$.

Let us prove that $q \sim p$ over $\psi$. Indeed let $\psi \to \pi$. If $\pi \in p$, then (since $p = q \cap r$) $\pi \in q$. Let $\pi \in q$. Since $\psi \in r$ and $\psi \to \pi$, $\pi \in r$. Thus (adding) we get $\pi \in p$.

It remains to check that $\varphi' \to \psi$. Since $\varphi'$ and $\psi$ are on the same line $v(x) + v(y) = v(a)$, they are comparable. If $\psi \to \varphi'$, then $\varphi' \in r$ and we can take $\psi = \varphi'$. $\square$

From this proposition it follows that $q$ and $r$ look like complementary sets of teeth for a saw.

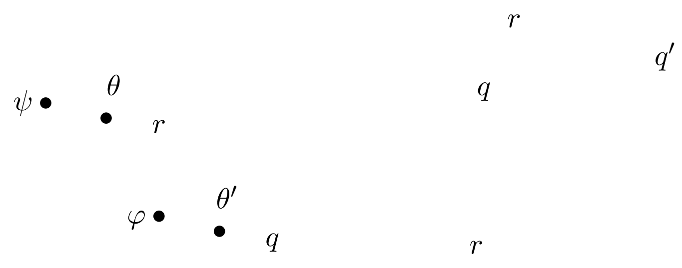


Figure 8:

**Proposition 5.2** *Let $N(p) = N(q) \oplus N(r)$. Then $p$, $q$ and $r$ are related as the graphs of the functions $f(p)$, $f(q)$ and $f(r)$ in Figure 7 on the right.*

**Proof.** It is clear that $p = q$ in a neighborhood of at least one pp-formula $\varphi$ (for instance one can take $x = 0 \in q^-$ and apply Lemma 5.1). We show that $\psi \in r$ for every pp-formula with $\psi \in p^-$ from this neighborhood of $q$. Indeed let us assume that $\psi \in r^-$. Then by Lemma 5.1 there exists $\xi \in r^- \cap q^+$ such that $\psi \to \xi$ and $r \sim p$ over $\xi$ (see Figure 7 on the left).

Then $\xi \in p^-$ which contradicts $q \sim p$ over $\varphi$. □

Let us return back to the example in Figure 4 on the right. For the (canonical) decomposition $N(p) = N(q') \oplus N(r)$ as shown in Figure 8 on the right we get $N(q') \cong N(q)$.

PP-types $p$ and $q$ are called *orthogonal* if $N(p)$ and $N(q)$ do not have an isomorphic nonzero direct summand.

**Corollary 5.3** *Let $N(p) = N(q) \oplus N(r)$. Then the modules $N(q)$ and $N(r)$ are orthogonal.*

**Proof.** It suffices to prove that for any such decomposition, $N(q)$ is not a direct summand of $N(r)$. Indeed if $N(q) = N(s) \oplus N(q')$ and $N(r) = N(s) \oplus N(r')$, then $N(p) = N(s) \oplus (N(s) \oplus N(q) \oplus N(r'))$.

Assume that $N(q)$ is isomorphic to a direct summand of $N(r)$. Since $x = 0 \in q^-$, by Lemma 5.1 there is $\varphi \in q^- \cap r$ such that $q \sim p$ over $\varphi$. Also by Proposition 4.1 we may assume that the $\varphi$-neighborhood of $q$ is isomorphic to a $\psi$-neighborhood of $r$, $\psi \in r^-$, along a line $v(x)+v(y) = v(a)$.

By Lemma 5.1 again, there is $\theta \in r^- \cap q$ such that $\psi \to \theta$ and $r \sim p$ over $\theta$. Let $\theta$ go to $\theta'$ under the translation identifying the $\psi$-neighborhood of $r$ and the $\varphi$-neighborhood of $q$ (see Figure 8 on the left), in particular $\theta' \in q^-$. Since $\varphi \to \theta'$, $\theta' \in r$. But the pp-formulae $\theta \in r^- \cap q$ and $\theta' \in q^- \cap r$ are on the same line, hence comparable, a contradiction. □

# 6 Corollaries

Recall that a von Neumann regular ring $S$ is called *abelian regular*, if all idempotents of $S$ are central. It is equivalent that $S$ be regular and (left and right) distributive. We say that the ring $S$ is *semiregular* if $S/\operatorname{Jac}(S)$ is regular and idempotents can be lifted modulo $\operatorname{Jac}(S)$. It has been mentioned above that the endomorphism ring $S$ of an arbitrary pure injective module (over any ring) is semiregular and $S/\operatorname{Jac}(S)$ is right self-injective. It is not true in general that $S/\operatorname{Jac}(S)$ is abelian regular since, for $T = \operatorname{End}(M \oplus M)$, $T/\operatorname{Jac}(T) = M_2(S/\operatorname{Jac}(S))$ is not.

**Lemma 6.1** *Let $p$ be a pp-type over a commutative valuation domain and $S = \operatorname{End}(N(p))$. Then $S/\operatorname{Jac}(S)$ is an abelian regular ring.*

**Proof.** Assume that $S' = S/\operatorname{Jac}(S)$ is not abelian regular. Then by [4, Thm. 3.4], there is a direct summand of the right $S'$-module $S'$ of the form $T' \oplus T'$. Since idempotents can be lifted, one can lift this decomposition to $S_S = T \oplus T \oplus U$. Since there is a 1-1 correspondence between direct sum decompositions of $S_S$ and $N(p)$, it follows that $N(p) = K \oplus K \oplus L$, a contradiction to Corollary 5.3. □

An idempotent $e$ of a von Neumann regular ring $S$ is called *abelian*, if the ring $eSe$ is abelian regular. A regular right self-injective ring $S$ is of *type I* if the twosided ideal generated by abelian idempotents is essential as a right ideal in $S$.

**Proposition 6.2** *Let $M$ be a pure injective module over a commutative valuation domain and $S = \operatorname{End}(M)$. Then $S' = S/\operatorname{Jac}(S)$ is a von Neumann regular right self-injective ring of type I and idempotents can be lifted modulo $\operatorname{Jac}(S)$.*

**Proof.** It remains to prove that $S'$ is of type I. If $0 \neq m \in M$ and $T = \operatorname{End}(N(m))$, then $T/\operatorname{Jac}(T)$ is abelian regular by Lemma 6.1. So we can apply [3, p. 33]. □

A module $M$ is called *Bezout* if every finitely generated submodule of $M$ is cyclic.

**Proposition 6.3** *Let $M$ be a pure injective module over a commutative valuation domain and $S = \mathrm{End}(M)$. Then ${}_SM$ is a Bezout module.*

**Proof.** By [6], ${}_SM$ is a distributive module. Also by [10, 3.33] every distributive module over a ring that is abelian regular modulo its radical is Bezout. So by Lemma 6.1 every module $N(p)$ is Bezout.

Let $m, n \in M$: we prove that $Sm + Sn \subseteq {}_SM$ is a cyclic module. Decompose $M = N(m) \oplus N$, where $m = (m, 0)$ and $n = (n_1, n_2)$. Also set $N = N(n_2) \oplus N'$, i.e. $N = N(m) \oplus N(n_2) \oplus N'$ with $m = (m, 0, 0)$, $n = (n_1, n_2, 0)$ in this decomposition. Since $m, n_1 \in N(m)$, there is $k_1 \in N(m)$ such that $eSem + eSen_1 = eSek_1$, where $e$ is the projection onto $N(m)$, in particular $Sm + Sn_1 = Sk_1$. Let $k = (k_1, n_2, 0)$. Then $\pi_1(k) = k_1$, $\pi_2(k) = n_2$, hence $m, n_1, n_2 \in Sk$ which yields $m, n \in Sk$. Since $k \in Sm + Sn$, $Sm + Sn = Sk$. $\square$

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# References

[1] P. Eklof P. and I. Herzog, *Some model theory over a serial ring*, Ann. Pure Appl. Logic, **72** (1995), 145–176.

[2] L. Fuchs and L. Salce, Modules over valuation domains, Lect. Notes Pure Appl. Math., Marcel Decker, **97** (1985).

[3] A. Facchini, *Decompositions of algebraically compact modules*, Pacific J. Math., **116**(1) (1985), 25–37.

[4] K. R. Goodearl, Von Neumann regular rings, Pitman, 1979.

[5] G.E. Puninski, *Superdecomposable pure injective modules over commutative valuation rings*, Algebra i Logika, **31**(6) (1992), 377–386.

[6] G.E. Puninski, *Indecomposable pure injective modules over chain rings*, Trans. Moscow Math. Soc., **56** (1995), 137–147.

[7] G. Puninski, Serial rings, manuscript of projected book, 1999.

[8] M. Prest, Model theory and modules. Cambridge University Press, London Math. Soc. Lecture Note Series, **130** (1988).

[9] L. Salce, *Valuation domains with superdecomposable pure injective modules*, Lect. Notes Pure Appl. Math., **146** (1993), 241–245.

[10] A.A. Tuganbaev, Semidistributive rings and modules, Mathematics and its applications, 449, Kluwer, 1998.

[11] M. Ziegler, *Model theory of modules*, Ann. Pure Appl. Logic, **26** (1984), 149–213.

[12] B. Zimmermann-Huisgen and W. Zimmermann, *Algebraically compact rings and modules*, Math. Z., **161** (1978), 81–93.