THE KRULL–GABRIEL DIMENSION OF A SERIAL RING

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Abstract. We prove that every serial ring R has the isolation property: every isolated point in any theory of modules over R is isolated by a minimal pair. Using this we calculate the Krull–Gabriel dimension of the module category over serial rings. For instance, we show that this dimension cannot be equal to 1.

1. Introduction

Let A be a finite dimensional algebra over a field $\Bbbk$, and let $\operatorname{mod}(A)$ be the category of finite dimensional A-modules. An object of many investigations in the representation theory is the category $\operatorname{mod}(\operatorname{mod}(A))$ of finitely presented additive covariant functors from $\operatorname{mod}(A)$ to the category of $\Bbbk$-vector spaces. For instance, the Krull–Gabriel dimension of this category, $\operatorname{KG}(A)$, is of special interest (see [14]).

It was noticed in Burke [1] that over a general ring R this notion splits into two parts: we may consider the Gabriel dimension of R, and its finitely presented variant, the Krull–Gabriel dimension of R. The choice depends on whether we factor out arbitrary simple functors or just those simple functors which are finitely presented in the relevant factor category.

These dimensions may be different, but, by [1, Thm. 5.1], they coexist.

In this paper we consider the Krull–Gabriel dimension, KG, of a serial ring. In particular we prove that for a serial ring R, $\operatorname{KG}(R)$ exists iff the lattice of right (left) ideals of R has Krull dimension (in the sense of Gabriel and Rentschler). Moreover, if the Krull dimension of R is equal to α, then $\operatorname{KG}(R)$ does not exceed $\alpha \oplus \alpha$, and we show that it is precisely $\alpha \oplus \alpha$ for some classes of serial rings. Here $\alpha \oplus \alpha$ stands for the Cantor's sum of α and α.

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We also prove that the Krull–Gabriel dimension of a serial ring can not be equal to 1. This result seems to be similar to the theorem of Krause [6] that for any finite dimensional algebra over an algebraically closed field, its Krull–Gabriel dimension is not equal to 1. Herzog [5] proved that this is true for any artin algebra.

All the results of this paper are tightly connected with the calculation of the Cantor–Bendixson rank of the Ziegler spectrum over a serial ring R, $\mathrm{CB}(\mathrm{Zg}_R)$, made by Reynders [13] and Puninski [10]. In fact we will prove that the so-called isolation property holds true over any serial ring R: every isolated point in the Ziegler spectrum of any theory of R-modules is isolated by a minimal pair.

An immediate consequence of this result is that the following invariants of a serial ring R are equal: 1) the Krull–Gabriel dimension of R; 2) the Cantor–Bendixson rank of the Ziegler spectrum over R; 3) the m-dimension of the lattice of all positive-primitive formulae over R. Having proved this result, instead of calculating the Krull–Gabriel dimension directly, we use the arithmetic of positive-primitive formulae over a serial ring developed in [12].

For instance we prove that if R is a semi-duo serial ring of (finite) Krull dimension n, then the Krull–Gabriel dimension of R is equal to $2n$.

2. Preliminaries

Let $\mathrm{mod}(R)$ be the category of finitely presented (right) modules over a ring R. By $F(R)$ we denote the category of covariant additive functors from $\mathrm{mod}(R)$ to the category of abelian groups, and let $\mathrm{mod}(\mathrm{mod}\,R)$ be the full subcategory of $F(R)$ consisting of finitely presented functors. It is well known that $\mathrm{mod}(\mathrm{mod}\,R)$ is an abelian category. As in [1] (see also [7]) we define the *Krull–Gabriel filtration* of $\mathrm{mod}(\mathrm{mod}\,R)$ as a non-decreasing sequence of Serre subcategories (the original definition of Geigle [2] uses contravariant functors).

Take $F_{-1} = \{0\}$ and recursively define F_α such that $F_{\alpha+1}$ consists of functors that are of finite length in the quotient category $\mathrm{mod}(\mathrm{mod}\,R)/F_\alpha$ (and $F_\lambda = \cup_{\mu<\lambda} F_\mu$ at limit stages). If at some stage α the localizing category generated by F_α is equal to $F(R)$, and α is minimal such, we say that the *Krull–Gabriel dimension* of R is equal to α.

For more of this, in particular how the Krull–Gabriel dimension is connected with the Gabriel dimension, the reader is referred to [1]. For instance,

$\mathrm{KG}(\mathbb{Z}) = 2$, but the Gabriel dimension of $\mathbb{Z}$ is equal to 1. Note that our terminology is the same as in Krause [7] or Prest [9].

The basic notions of the model theory of modules, including the Ziegler spectrum, may be found in Prest's book [8]. For instance, $pp_M(m)$ will denote the pp-type of an element m of a module M, i.e. the set of all pp-formulae which are satisfied by m in M.

Let φ, ψ be pp-formulae over a ring R. We will use (φ/ψ) to denote, depending on the context, 1) the interval $[\varphi \wedge \psi, \varphi]$ in the lattice of all pp-formulae over R; 2) the pair of pp-formulae (φ, ψ) (usually with $\psi < \varphi$), or 3) the basic open set (φ/ψ) in the Ziegler spectrum Zg_R over R.

Let T be a theory of modules over a ring R. A pair of pp-formulae (φ, ψ) is called *minimal*, if the interval (φ/ψ) is simple in T. We say that T has *the isolation property*, if for any extension T' of T, every isolated point in $\mathrm{Zg}_{T'}$ is isolated by a minimal pair.

Recall that a module M is called *distributive*, if the lattice of submodules of M is distributive. We say that M is *endo-distributive*, if M is distributive as a module over its endomorphism ring $S = \mathrm{End}(M)$. Similarly we say that a module M is *pp-distributive*, if the lattice of pp-definable subgroups of M is distributive.

A module M is said to be *uniserial*, if the lattice of submodules of M is a chain, and M is *serial*, if M is a direct sum of uniserial modules. For instance, every uniserial module is distributive.

A ring R (with a unit 1) is *serial*, if R_R is a serial right R-module and ${}_RR$ is a serial left R-module. Thus R is serial iff there exists a collection $e_1, \ldots, e_n$ of orthogonal idempotents such that 1) $1 = e_1 + \cdots + e_n$; 2) every right module e_iR is uniserial, and every left module Re_i is uniserial. Usually we consider a serial ring R with a fixed system of orthogonal (indecomposable) idempotents $e_1, \ldots, e_n$.

If R is a serial ring, then every 'diagonal' ring $R_i = e_iRe_i$ is uniserial, and e_iRe_j is an R_i-R_j-bimodule. Let us define a non-increasing sequence $\mathrm{Jac}(\alpha)$ of two-sided ideals of R. Put $\mathrm{Jac}(0) = \mathrm{Jac}(R)$, and $\mathrm{Jac}(\alpha+1) = \cap_n \mathrm{Jac}^n(\alpha)$ at non-limit stages. If λ is limit then set $\mathrm{Jac}(\lambda) = \cap_{\mu<\lambda} \mathrm{Jac}(\mu)$.

By Müller (see [11, Prop. 1.30]) the Krull dimension of a serial ring R, $\mathrm{Kdim}(R)$, is equal to α iff α is the least ordinal such that the ideal $\mathrm{Jac}(\alpha)$ is a nilpotent. In particular, the right Krull dimension of R is equal to the left Krull dimension of R.

3. The isolation property

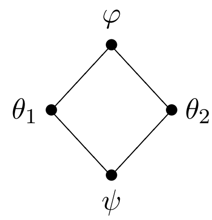
Theorem 3.1. *Let T be a theory of modules, and suppose that every indecomposable pure-injective module M in Zg_T is contained in a basic open set (φ/ψ), such that the interval (φ/ψ) in the lattice of pp-definable subgroups of M has width. Then the isolation property holds for T.*

Proof. Since the above condition is inherited by extensions of T, it suffices to prove that every isolated point in Zg_T is isolated by a minimal pair.

Let M be an isolated point of Zg_T. By hypothesis, M is contained in a basic open set (φ/ψ) (with $\psi < \varphi$), such that the lattice of pp-definable subgroups of M between $\psi(M)$ and $\varphi(M)$ has width. Then this lattice contains a subinterval that is a chain, so we may assume that the interval $[\psi(M), \psi(M)]$ is a chain from very beginning.

By [16, 4.9] a basis of neighborhoods for M in Zg_T can be chosen from the collection of subpairs of the pair (φ/ψ). Thus we may assume that (φ/ψ) isolates M in Zg_T.

Now we prove that the lattice of pp-formulae in T between ψ and φ is a chain. Otherwise there are incomparable pp-formulae θ_1, θ_2 between ψ and φ in T.



So both pairs (θ_1/θ_2) and (θ_2/θ_1) are nontrivial in T. Since they are subpairs of (φ/ψ), they both must isolate M. But, by the choice of (φ/ψ), either $\theta_1(M) \subseteq \theta_2(M)$ or $\theta_2(M) \subseteq \theta_1(M)$ holds, hence at least one of these pairs is closed on M, a contradiction.

Thus the lattice of pp-subgroups in T between ψ and φ is a chain. By [8, Th. 10.2] the pair (φ/ψ) generates on every super-decomposable pure-injective module in T. From the remark in [8, p. 213] we conclude that M is isolated by a minimal pair in Zg_T. □

Proposition 3.2. *Let T be any theory of modules over a serial ring R. Then the isolation property holds for T.*

Proof. Let M be an indecomposable pure-injective R-module. Choose $0 \neq m \in M$, hence $me_i \neq 0$ for some i. Then M is in the basic open set

$(e_i \mid x/x = 0)$, where $e_i \mid x$ denotes the pp-formula 'e_i divides x'. But, by [11, L. 11.4], the lattice of pp-definable subgroups of M below Me_i is a chain.

Thus it remains to apply Theorem 3.1. □

Note that (see [11, Ch. 12]) there is a serial ring with a super-decomposable pure-injective module. Therefore Proposition 3.2 applies beyond the case when the continuous part of T is zero.

The theory T is called *distributive* if the lattice of all pp-formulae of T is distributive.

The following is a characterization of these theories.

Proposition 3.3. *For a theory T (over an any ring) the following are equivalent:*

1) T is distributive;

2) every pure-injective model of T is endo-distributive (pp-distributive);

3) every pure-injective indecomposable module in T is endo-uniserial (pp-uniserial).

Proof. 1) ⇒ 2). Let M be a pure-injective model of T, $S = \mathrm{End}(M)$. We prove that M is a distributive left S-module. By [15, 1.15 (iii)] it suffices to check the distributivity on cyclic S-submodules Sm, $m \in M$. Since M is pure-injective, every such submodule is of the form $p(M)$, where $p = pp_M(m)$.

So let $m, n, k \in M$, $p = pp_M(m)$, $q = pp_M(n)$ and $r = pp_M(k)$. It is enough to check the inclusion

$$p(M) \cap (q(M) + r(M)) \subseteq (p(M) \cap q(M)) + (p(M) \cap r(M)).$$

Let $l \in p(M) \cap (q(M) + r(M))$, i.e. $l \in (\varphi \wedge (\psi + \theta))(M)$ for any $\varphi \in p$, $\psi \in q$, and $\theta \in r$. Since T is distributive, it follows that $l \in ((\varphi \wedge \psi) + (\varphi \wedge \theta))(M)$. But M is pure-injective. Then [8, Cor. 2.3] yields $l \in (p(M) \cap q(M)) + (p(M) \cap r(M))$.

Since every endo-distributive module has a distributive lattice of pp-subgroups, M is pp-distributive.

2) ⇒ 1) is clear, because the lattice of pp-formulae of T coincides with the lattice of pp-subgroups of a 'large' pure-injective model of T.

2) ⇒ 3). Let M be a pure-injective indecomposable model of T, $S = \mathrm{End}(M)$. By hypothesis ${}_SM$ is distributive. Since S is a local ring, by Stephenson (see [15, Th. 2.1]) ${}_SM$ is a uniserial module.

3) ⇒ 2). Let M be a pure-injective model of T. By [8, Cor. 3.8] M is a direct summand of a direct product of indecomposable pure-injective models of T. But by [15, 8.6 (i)] endo-distributivity preserves under direct products and direct summands. □

Proposition 3.4. *The isolation property holds for every distributive theory.*

Proof. By Proposition 3.3 and Theorem 3.1. □

The standard example of a distributive theory is the theory of all modules over a commutative Prüfer ring. Thus we obtain the following.

Corollary 3.5. *The isolation property holds for every theory of modules over a commutative Prüfer ring.*

Since we are mainly working with serial rings, let us derive some standard conclusions only in this case.

Proposition 3.6. *Let R be a serial ring. Then the following invariants of R are equal:*

1) the Cantor–Bendixson rank of the Ziegler spectrum of R;

2) the m-dimension of the lattice of all pp-formulae over R;

3) the Krull–Gabriel dimension of R.

Proof. By Proposition 3.2, the isolation property holds for every theory of modules over a serial ring R. Thus 1) and 2) are equivalent by [8, Prop. 10.19]. The remaining equivalence is proved in [1, Thm. 4.5]. □

4. m-dimension

If a, b are elements of a lattice L, (a/b) will denote the interval $[a \wedge b, b]$.

Let L be a lattice with $0 \neq 1$. We define a non-decreasing sequence of congruences $\sim_\alpha$ on L. Let $\sim_{-1}$ be the trivial congruence. If $\sim_\alpha$ has already been defined, put $a \sim_{\alpha+1} b$ if the interval (a/b) has finite length in the lattice $L_\alpha = L/\sim_\alpha$. At a limit stage λ we set $\sim_\lambda = \cup_{\mu<\lambda} \sim_\mu$.

The m-dimension of L, $\mathrm{mdim}(L)$, is equal to α, if α is the least ordinal such that L_α consists of one element. For instance, $\mathrm{mdim}(L) = 0$ iff L is finite, and $\mathrm{mdim}(L)$ is not defined iff L contains the ordered set of the rationals $\mathbb{Q}$ as a sublattice.

We say that the interval (a/b) is *α-simple*, if $\mathrm{mdim}(a/b) = \alpha$ and for every $c \in (a/b)$, either $\mathrm{mdim}(a/c) < \alpha$ or $\mathrm{mdim}(c/b) < \alpha$ holds. Then

$\mathrm{mdim}(a/b) = \alpha$ iff (a/b) can be decomposed into finitely many α-simple intervals.

Lemma 4.1. *Let L be a lattice with $0 \neq 1$ and let $a \in L$. Then the m-dimension of L is equal to the maximum of the m-dimensions of the intervals $(a/0)$ and $(1/a)$.*

Proof. Let $\mathrm{mdim}(L) = \alpha$, $\mathrm{mdim}(a/0) = \beta$, and $\mathrm{mdim}(1/a) = \gamma$.

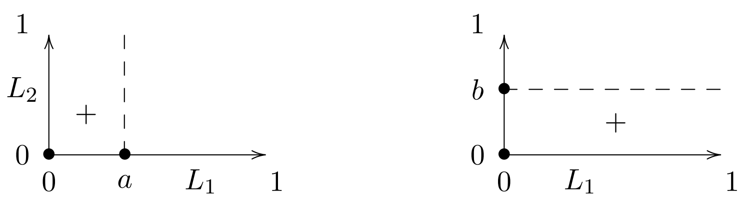
From $(a/0), (1/a) \subseteq L$ it follows that $\beta, \gamma \leq \alpha$, hence $\max(\beta, \gamma) \leq \alpha$.

Let us prove that $\alpha \leq \max(\beta, \gamma)$. Note that, if L' is a convex sublattice of L, then any congruence $\sim_\delta$ defined on L' is the restriction of the congruence $\sim_\delta$ defined on L. In particular this applies to the intervals $(a/0)$ and $(1/a)$.

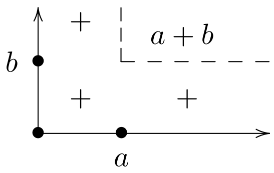
We may assume that $\beta \leq \gamma$. From $\mathrm{mdim}(a/0) = \beta$ it follows that $0 \sim_\beta a$ in $(a, 0)$, hence $a \sim_\beta 0$ in L. Then $a \sim_\gamma 0$ in L, and similarly $a \sim_\gamma 1$ in L. Thus $0 = 1$ in $L/\sim_\gamma$, hence $\mathrm{mdim}(L) \leq \gamma$. □

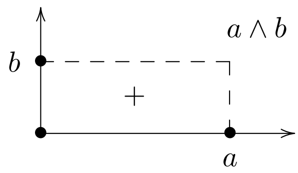
Let L_1 and L_2 be chains with $0 \neq 1$. By $L = L_1 \otimes L_2$ we will denote the modular lattice freely generated by L_1 and L_2 with respect to the relations $0_1 = 0_2$ and $1_1 = 1_2$ (i.e. the smallest and the largest elements of L_1 and L_2 are identified). For instance, if L_2 consists of two elements, then $L_1 \otimes L_2 = L_1$.

It is well known (see [4, Thm. 13]) that this lattice is distributive. Moreover it is quite easy to describe the shape of elements of this lattice. Let us represent elements $a \in L_1$ and $b \in L_2$ by rectangles (in the plane $L_1 \times L_2$) in the following way:

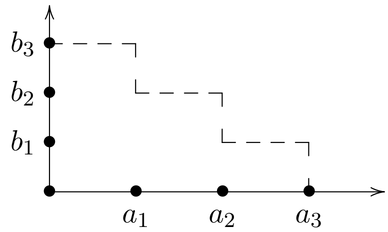


Then L is isomorphic to the lattice of subsets of the plane $L_1 \times L_2$ generated by these rectangles with respect to usual set theoretic operations $\cap$ and $\cup$. For instance the resulting figures for elements $a + b$ and $a \wedge b$ are the following:

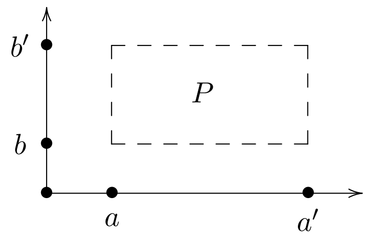




Thus every element of L can be (uniquely) represented as $(a_n \wedge b_1) + (a_{n-1} \wedge b_2) + \cdots + (a_1 \wedge b_n)$, where $a_1 < a_2 < \cdots < a_n \in L_1$ and $b_1 < b_2 < \cdots < b_n \in L_2$, i.e. as a descending ladder with just finitely many steps (some of the steps may be infinite). For instance, if $a_1 < a_2 < a_3 \in L_1$ and $b_1 < b_2 < b_3 \in L_2$, then $(a_3 \wedge b_1) + (a_2 \wedge b_2) + (a_1 \wedge b_3)$ looks as follows:



Given $a < a' \in L_1$ and $b < b' \in L_2$, the interval $(a' \wedge b'/a + b)$ can be thought as the rectangle $P = (a'/a) \times (b'/b)$:

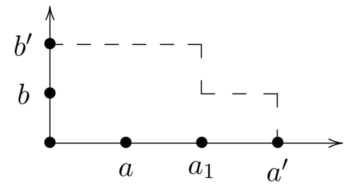


If α, β are ordinals, then $\alpha \oplus \beta$ will denote their Cantor sum. To calculate this, just represent $\alpha = \omega^{\alpha_1} n_1 + \cdots + \omega^{\alpha_k} n_k$ and $\beta = \omega^{\alpha_1} m_1 + \cdots + \omega^{\alpha_k} m_k$, where $\alpha_1 > \alpha_2 > \cdots > \alpha_k$ (some of n_i, m_i may be zero). Then $\alpha \oplus \beta = \omega^{\alpha_1}(n_1 + m_1) + \cdots + \omega^{\alpha_k}(n_k + m_k)$. In particular $\alpha \oplus \beta = \beta \oplus \alpha$.

Proposition 4.2. *Let L_1 and L_2 be chains with $0 \neq 1$, and $L = L_1 \otimes L_2$. Then* $\mathrm{mdim}(L) = \mathrm{mdim}(L_1) \oplus \mathrm{mdim}(L_2)$.

Proof. Let $a < a' \in L_1$, $b < b' \in L_2$, and let P be the rectangle $(a'/a) \times (b'/b)$, i.e. the interval $(a' \wedge b'/a + b)$ in L. By induction on γ we prove that P is γ-simple in L iff (a'/a) is α-simple in L_1, (b'/b) is β-simple in L_2, and $\gamma = \alpha \oplus \beta$.

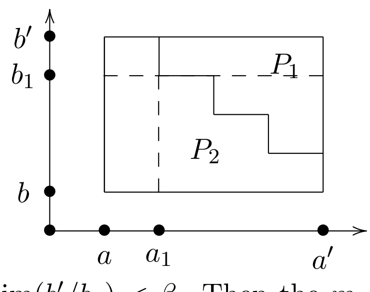
As a basis of the induction take $\gamma = 0$. If (a'/a) is simple (i.e. 0-simple) in L_1 and (b'/b) is simple in L_2 then by the above description P is simple in L. Suppose that P is simple in L. If (a'/a) is not simple in L_1 then there exists $a_1 \in L_1$ such that $a < a_1 < a'$. Clearly $(a' \wedge b) + (a_1 \wedge b')$ is between $a' \wedge b'$ and $a' \wedge b' \wedge (a + b) = (a \wedge b') + (a' \wedge b)$, a contradiction.



Thus we may assume that we have already established the description of δ-simple rectangles P in L for every $\delta < \gamma$. Let $P = (a'/a) \times (b'/b)$ be such that $\mathrm{mdim}(a'/a) = \alpha'$, $\mathrm{mdim}(b'/b) = \beta'$, and $\alpha' \oplus \beta' < \gamma$. Then P is decomposed into finitely many rectangles whose sides are δ-simple for certain $\delta < \alpha \oplus \beta$. By Lemma 4.1 we obtain $\mathrm{mdim}(P) = \alpha' \oplus \beta'$.

Let us assume that (a'/a) is α-simple in L_1, (b'/b) is β-simple in L_2, where $\alpha \oplus \beta = \gamma$. We prove that P is $\gamma = \alpha \oplus \beta$-simple (it is not γ'-simple for every $\gamma' < \alpha \oplus \beta$ by the induction hypothesis). Otherwise P can be decomposed as $P_1 \cup P_2$ such that both figures P_1 and P_2 have m-dimension not less than γ.

The case when P is sectioned only by one horizontal or one vertical line is clear. Suppose that the board of P_1 and P_2 is a nontrivial ladder as shown below. We use an additional induction on the number of steps of this ladder.



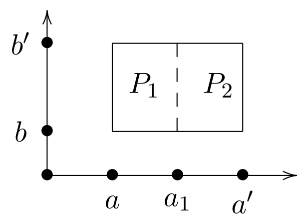
Suppose that $\mathrm{mdim}(b'/b_1) < \beta$. Then the m-dimension of the rectangle $P' = (a'/a) \times (b'/b_1)$ is less than γ. Also the interval (b_1/b) is β-simple. Thus we may remove P' from P_1 and P_2 without changing the m-dimension. Since we have decreased the number of steps, the result will follow by induction.

Otherwise $\mathrm{mdim}(b'/b_1) = \beta$. Since the interval (b'/b) is β-simple, it follows that $\mathrm{mdim}(b_1/b) < \beta$. Suppose that $\mathrm{mdim}(a'/a_1) < \alpha$. Then the

m-dimension of the rectangle $(a'/a_1) \times (b'/b)$ is less than γ by induction, hence $\mathrm{mdim}(P_1)$ is less than γ (since P_1 is contained in this rectangle), a contradiction.

Thus we may assume that $\mathrm{mdim}(a'/a_1) = \alpha$. Since (a'/a) is α-simple, it follows that $\mathrm{mdim}(a_1/a) < \alpha$. Then (by induction) both rectangles $(a_1/a) \times (b'/b)$ and $(a'/a_1) \times (b_1/b)$ have m-dimension less than γ. Since P_2 is contained in the union of these rectangles, we conclude that $\mathrm{mdim}(P_2) < \gamma$, a contradiction.

So it remains to prove that there are no other simple rectangles P with dimension $\alpha \oplus \beta$. First suppose that $P = (a'/a) \times (b'/b)$, $\mathrm{mdim}(a'/a) = \alpha'$, $\mathrm{mdim}(b'/b) = \beta'$ and $\alpha' \oplus \beta' > \gamma = \alpha \oplus \beta$. By symmetry we may assume that there is $\gamma' < \alpha'$ such that $\gamma' \oplus \beta' \geq \gamma$. Since $\mathrm{mdim}(a'/a) > \gamma'$, this interval is not γ'-simple. Therefore there exists $a_1 \in (a'/a)$ such that $\mathrm{mdim}(a_1/a) \geq \gamma'$ and $\mathrm{mdim}(a'/a_1) \geq \gamma'$. Let us split $P = P_1 \cup P_2$ in the following way



Then, by induction, $\mathrm{mdim}(P_1) \geq \gamma$ and $\mathrm{mdim}(P_2) \geq \gamma$, hence P is not γ-simple.

It remains to consider the case when $\alpha' \oplus \beta' = \alpha \oplus \beta$ but one of segments, say (a'/a) is not α'-simple. Then there is $a_1 \in (a'/a)$ such that $\mathrm{mdim}(a_1/a) = \mathrm{mdim}(a'/a_1) = \alpha'$. Decomposing P as above we obtain that P is not γ-simple.

As the final step we deduce that the m-dimension of the rectangle $(1_1/0_1) \times (1_2/0_2)$ is equal to $\mathrm{mdim}(L_1) \oplus \mathrm{mdim}(L_2)$. But this rectangle is the interval $(1_1 \wedge 1_2/0_1 + 0_2) = (1/0) = L$. □

By $L_1 \otimes' L_2$ let us denote the modular lattice freely generated by L_1 and L_2 (without the identification of 0 and 1).

Corollary 4.3. *Let L_1, L_2 be chains with $0 \neq 1$ and $L = L_1 \otimes' L_2$. Then* $\mathrm{mdim}(L) = \mathrm{mdim}(L_1) \oplus \mathrm{mdim}(L_2)$.

Proof. It is clear that the lattice $L_1 \otimes L_2$ is a factor lattice of L, therefore $\mathrm{mdim}(L) \geq \mathrm{mdim}(L_1 \otimes L_2) = \mathrm{mdim}(L_1) \oplus \mathrm{mdim}(L_2)$.

For the converse let L'_1 be obtained from L_1 by adding a new smallest element 0_1^- and a new largest element 1_1^+, and similarly for L_2. Since $0 \neq 1$ in L_1 and L_2 we have $\mathrm{mdim}(L_1) = \mathrm{mdim}(L'_1)$ and $\mathrm{mdim}(L_2) = \mathrm{mdim}(L'_2)$. If $L' = L'_1 \otimes L'_2$ then $\mathrm{mdim}(L') = \mathrm{mdim}(L'_1) \oplus \mathrm{mdim}(L'_2) = \mathrm{mdim}(L_1) \oplus \mathrm{mdim}(L_2)$. But clearly $L_1 \otimes' L_2$ is a sublattice of L'. □

5. Krull–Gabriel dimension

In this section we investigate the Krull–Gabriel dimension of a serial ring.

Recall that the *Krull dimension* of a lattice L, $\mathrm{Kdim}(L)$, is defined as follows. $\mathrm{Kdim}(L) = 0$ iff L has the descending chain condition, i.e. if L is artinian. By induction on ordinals we define $\mathrm{Kdim}(L) \geq \alpha + 1$, if there is a descending chain $a_1 > a_2 > \dots$, $a_i \in L$ such $\mathrm{KG}(a_{i+1}/a_i) \geq \alpha$ for each i. For a limit ordinal γ we set $\mathrm{Kdim}(L) \geq \gamma$ if $\mathrm{Kdim}(L) \geq \beta$ for every $\beta < \gamma$.

Now the Krull dimension of L is the smallest ordinal α such that $\mathrm{Kdim}(L) \geq \alpha + 1$ fails.

In general the Krull dimension and the m-dimension of a lattice L may differ drastically (although they coexist). For example, if $L = \omega^\alpha + 1$, then $\mathrm{Kdim}(L) = 0$ but $\mathrm{mdim}(L) = \alpha$. But for a serial ring they are almost the same.

Recall that an idempotent e of a ring R is said to be *indecomposable*, if the projective right module eR is indecomposable. If R is a serial ring, then e is indecomposable iff eR is a uniserial module.

Lemma 5.1. *Let e be an indecomposable idempotent of a serial ring R. Then the following invariants are equal:*

1) the Krull dimension of the lattice of all submodules of eR;

2) the Krull dimension of the lattice of cyclic submodules of eR;

3) the m-dimension of the lattice of cyclic submodules of eR.

Proof. Let L be the lattice of all submodules of eR, and let L' be the lattice of all cyclic submodules of eR. Since L' is a sublattice of L, $\mathrm{Kdim}(L') \leq \mathrm{Kdim}(L)$.

Let us prove that $\mathrm{Kdim}(L') \geq \mathrm{Kdim}(L)$, hence $\mathrm{Kdim}(L') = \mathrm{Kdim}(L)$. If $\mathrm{Kdim}(L) \geq \alpha$, then there exists an order inverting embedding $f : \omega^\alpha \to L$. If $\beta \in \omega^\alpha$, then $f(\beta) \supset f(\beta + 1)$ yields that $r_\beta \in f(\beta) \setminus f(\beta + 1)$ for some $r_\beta \in R$. Thus $f(\beta) \supseteq r_\beta R \supset f(\beta + 1) \supseteq r_{\beta+1} R$ implies $r_\beta R \supset r_{\beta+1} R$. Therefore the map $\beta \to r_\beta R$ defines an order inverting embedding g from ω^α to L'. Thus $\mathrm{Kdim}(L') \geq \alpha$.

Let us extend g to $\omega^\alpha + 1$ by sending 1 to the zero ideal. Since $\mathrm{mdim}(\omega^\alpha + 1) = \alpha$ we obtain $\mathrm{mdim}(L') \geq \alpha$.

It remains to prove that $\mathrm{mdim}(L') \leq \mathrm{Kdim}(L)$. If $\mathrm{mdim}(L')$ does not exist then there is a countable dense subset in L', hence $\mathrm{Kdim}(L)$ does not exist. So we may assume that $\mathrm{Kdim}(eR)$ is defined.

By [11, Prop. 1.30] $\mathrm{Kdim}(eR)$ is the least α such that $e\,\mathrm{Jac}(\alpha)^n = 0$ for some n. By induction on $\beta \leq \alpha$ we prove that every interval (rR/rsR) for $r \in eR$, $s \in \mathrm{Jac}(\beta)^k$ has m-dimension not more than β. Decomposing s in a product we may assume that $s \in \mathrm{Jac}(\beta) \setminus \mathrm{Jac}(\beta)^2$. If $\mathrm{mdim}(rR/rsR) > \beta$ then there is a principal ideal $I \subseteq eR$ such that $rsR \subset I \subset rR$ and $\mathrm{mdim}(I/rsR) \geq \beta$, $\mathrm{mdim}(rR/I) \geq \beta$. Clearly $I = rtR$ where $s = ts'$ for some $t, s' \in R$. Since $s \notin \mathrm{Jac}(\beta)^2$ by symmetry we may assume that $t \notin \mathrm{Jac}(\gamma)^l$ for some l and $\gamma < \beta$. Then, by induction, $\mathrm{mdim}(rR/rtR) \leq \gamma$, a contradiction. □

The following lemma gives a neccesary condition for the existence of the Krull–Gabriel dimension of a serial ring.

Lemma 5.2. *Let R be a serial ring. If the Krull–Gabriel dimension of R exists then the Krull dimension of R is defined.*

Proof. Suppose that the (left) Krull dimension of R is undefined. Then for some i the left module Re_i does not have Krull dimension. Therefore by Lemma 5.1 there exists a dense chain of principal submodules of Re_i.

Thus there are $r_q \in Re_i$, $q \in \mathbb{Q}$ such that $r_q \in Rr_{q'}$ iff $q \geq q'$. Let φ_q, $q \in \mathbb{Q}$ be divisibility formulae $r_q \mid x$. Clearly $\varphi_q \to \varphi_{q'}$ iff $q \geq q'$, hence this chain of pp-formulae is dense. Thus the m-dimension of the lattice of all pp-formulae over R does not exist. By Proposition 3.6 the Krull–Gabriel dimension of R does not exist. □

So in further considerations we restrict ourselves on the case of serial rings with Krull dimension.

Now we find an upper bound for Krull–Gabriel dimension of a serial ring. This result is very similar to considerations in Reynders [13, Cor. 3.27] concerning Cantor–Bendixson rank (there is a small lapse in that paper which is originated in the author's paper [10] — instead of 2α there should be $\alpha \oplus \alpha$ everywhere).

Proposition 5.3. *Let R be a serial ring with Krull dimension α. Then the Krull–Gabriel dimension over R is less than or equal to $\alpha \oplus \alpha$.*

Proof. Let L be the lattice of all pp-formulae over R. By Proposition 3.6 it suffices to prove that $\mathrm{mdim}(L) \le \alpha \oplus \alpha$.

By [11, L. 11.1] every interval $(e_1 \mid x/x = 0)$ in L is generated by two chains: L_1 consisting of divisibility formulae $a \mid x$, $a \in Re_1$, and L_2 consisting of annihilator formulae $xb = 0$, $b \in e_1R$. Since L_1 is anti-isomorphic to the lattice of cyclic submodules of Re_i, $\mathrm{mdim}(L_1) \le \alpha$. Also L_2 is isomorphic to the lattice of cyclic submodules of e_1R, which gives $\mathrm{mdim}(L_2) \le \alpha$.

Then $\mathrm{mdim}(e_1 \mid x/x = 0) \le \alpha \oplus \alpha$ by Proposition 4.2. By Lemma 4.1 it remains to prove that $\mathrm{mdim}(x = x/e_1 \mid x) \le \alpha \oplus \alpha$. But this interval is isomorphic to the interval $(e \mid x/x = 0)$ where $e = e_2 + \cdots + e_n$, so we can complete the proof by induction. □

Corollary 5.4. *Let R be a serial ring. Then the Krull–Gabriel dimension of R exists iff the Krull dimension of R is defined.*

Proof. By Lemma 5.2 and Proposition 5.3. □

Note that the above-mentioned bound for the Krull–Gabriel dimension of a serial ring is optimal. For instance (see [10, Cor. 3.6]), if V is a commutative valuation domain of Krull dimension α, then the CB-rank of Zg_V is equal to $\alpha \oplus \alpha$. Hence the Krull–Gabriel dimension of V is equal to $\alpha \oplus \alpha$.

Now we obtain a lower bound for the Krull–Gabriel dimension of a serial ring.

Let e be an indecomposable idempotent of a serial ring R. For $a, b \in Re$ put $a \le_l b$ if $b \in Ra$ (i.e. $Rb \subseteq Ra$), and $a <_l b$ if $Rb \subset Ra$. Similarly for $c, d \in eR$ we set $c \le_r d$ if $d \in cR$, and $c <_r d$ if $dR \subset cR$. Finally for $e, f \in R$ we write $e \le f$ if $f \in ReR$, and $e < f$ if $RfR \subset ReR$. Note that $<_l$ and $<_r$ are linear orders whereas $<$ is just a partial order.

Let L be a linear order with $0 \ne 1$, and let e be an indecomposable idempotent of a serial ring R. We say that the functions $f : L \to Re$, $g : L \to eR$ form a *boundary pair*, if for every $a, b \in L$, $a < b$ we have $f(a) <_l f(b)$, $g(a) <_r g(b)$ and $f(a)g(a) < f(b)g(b)$.

Proposition 5.5. *Let R be a serial ring, and let $f : L \to Re$, $g : L \to eR$ be a boundary pair. If the m-dimension of L is equal to α, then the Krull–Gabriel dimension of R is not less than $\alpha \oplus \alpha$.*

Proof. By Corollary 4.3 (and Proposition 3.6) it suffices to embed $L \otimes' L$ into the lattice P of all pp-formulae over R.

Note that for $l, l' \in L$ we have $f(l) \mid x \to f(l') \mid x$ iff $l' \leq l$, and $xg(l) = 0 \to xg(l') = 0$ iff $l \leq l'$. Thus there are two copies of L inside P: one is given by divisibility formulae $\varphi_l(x) =$ '$f(l) \mid x$', and the other is given by annihilator formulae $\psi_l(x) =$ '$xf(l) = 0$'. Both these chains have m-dimension α.

It remains to prove that the lattice of pp-formulae generated by these two chains is freely generated. Otherwise for some $l < l' \in L$ we have $f(l) \mid x \wedge xg(l') = 0 \to f(l') \mid x + xg(l) = 0$. By [12, L. 3.1] it follows that $f(l')g(l') \leq f(l)g(l)$, a contradiction. □

It is known by Krause [6] and Herzog [5] that there exists no finite dimensional algebra A such that the Krull–Gabriel dimension of A is 1. The corresponding question for $\mathrm{Zg}(A) = 1$ is still open. In the following corollary we consider this property for serial rings.

Corollary 5.6. *Let R be a serial ring. Then each of the following invariants is not equal to* 1*: 1) the Krull–Gabriel dimension of R; 2) the Cantor–Bendixson rank of the Ziegler spectrum of R; 3) the m-dimension of the lattice of all pp-formulae over R.*

Proof. We may assume that R has Krull dimension. If R is artinian then it has a finite representation type, hence all these dimensions are zero.

So we may assume that R is not artinian. Then (see [11, L. 1.32]) there exists an indecomposable idempotent e such that the uniserial ring eRe is not artinian. Thus we may assume that R is a uniserial non-artinian ring with Krull dimension.

Since R has Krull dimension, there exists $p \in R$ such that $\mathrm{Jac}(R) = pR = Rp$.

By Proposition 5.5 it suffices to construct a boundary pair $f, g : \omega{+}1 \to R$. Let us define $f(n) = g(n) = p^n$ for $n \in \omega$ and $f(1) = 0$. If $m < n \in \omega$ then clearly $f(m) = p^m <_l p^n = f(n)$ and similarly for g. Also $f(m)g(m) = p^{2m}$ and $f(n)g(n) = p^{2n}$. Clearly the assumption $p^{2m} \in Rp^{2n}R = p^{2n}R$ leads to a contradiction. □

Now it is easy to calculate the Krull–Gabriel dimension for some classes of serial rings.

Corollary 5.7. *Let R be a serial ring of Krull dimension* 1*. Then the Krull–Gabriel dimension of R is equal to* 2*.*

Proof. By Corollary 5.6, the Krull–Gabriel dimension of R is greater than 1. It remains to apply Proposition 5.3. □

According to [11, Ch. 7], every right noetherian (non-artinian) serial ring R has Krull dimension 1. Thus the Krull–Gabriel dimension of R is equal to 2. Note that Generalov [3] proved that the right Gabriel dimension of a right noetherian serial ring is equal to 1.

A serial ring R is called *semi-duo*, if for every $r \in e_i R e_j$ we have either $e_i R r \subseteq rR$ or $rRe_j \subseteq Rr$. By [11, Thm. 2.13] a serial ring R is semi-duo iff every finitely presented indecomposable (left or right) R-module has a local endomorphism ring.

Proposition 5.8. *Let R be a uniserial semi-duo ring of Krull dimension α. Then the Krull–Gabriel dimension of R is equal to $\alpha \oplus \alpha$.*

Proof. By Proposition 5.3 we obtain $\mathrm{KG}(R) \leq \alpha \oplus \alpha$.

Let $R' = R/\operatorname{Jac}(\alpha)$. By [12, Cor. 9.4] R' is a semi-duo uniserial domain of Krull dimension α. Then by [13, Cor. 3.27] the Cantor–Bendixson rank of $\mathrm{Zg}_{R'}$ is equal to $\alpha \oplus \alpha$. It remains to apply Proposition 3.6. □

For a serial semi-duo ring we have the following weaker form of the previous proposition.

Corollary 5.9. *Let R be a serial semi-duo ring of finite Krull dimension n. Then the Krull–Gabriel dimension of R is equal to $2n$.*

Proof. Every 'diagonal' ring $R_i = e_i R e_i$ is a uniserial semi-duo ring. Moreover, by [11, L. 1.32], there exists i such that the Krull dimension of R_i is equal to n. By Proposition 5.8 the Krull–Gabriel dimension of R_i is equal to $2n$. Then $\mathrm{KG}(R) \geq 2n$.

It remains to apply Proposition 5.3. □

Cornjecture 5.10. *Let R be a serial ring of finite Krull dimension. Then the Krull–Gabriel dimension of R is even.*

It is clear from [12] that the structure of the lattice of two-sided ideals of a serial ring should play a crucial role in calculations of Krull–Gabriel dimension.

Remark 5.11. *Let R be a serial ring. Then the lattice of two-sided ideals of R is distributive.*

Proof. If I is a two-sided ideal of R, $I = \oplus_{i,j} e_i I e_j$ (as an abelian group). It remains to note that $e_i R e_j$ is uniserial as a right $e_j R e_j$-module (or left $e_i R e_i$-module). □

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