SOLUTION OF APPLIED PROBLEMS: FORMALIZATION, METHODOLOGY, JUSTIFICATION

KRASNOPROSHIN VICTOR

Belarus State University, 4 Nezavisimosti Avenue, Minsk, 220050 Belarus, e-mail: krasnoproshin@ bsu.by

OBRAZTSOV VLADIMIR

Belarus State University, 4 Nezavisimosti Avenue, Minsk, 220050 Belarus, e-mail: obraztsov@bsu.by

VISSIA HERMAN

Byelex Multimedia Products BV Argon 1 – 4751 XC Oud Gastel, the Netherlands, e-mail: h.vissia@byelex.com

The paper deals with questions of management of the applied problem solution. Three groups of problems have been considered. The problems concern the formalization, selection and construction of the model, method and algorithm of the solution as well as the justification of the obtained results.

1. Introduction

With the development of the society, the role of theoretical knowledge and computer technology in solving applied problems is constantly rising. However, the theory and practice are developing in some sense independently of each other. Each of them has its own specific features and priorities. Using the theory for practice usually occurs on formal grounds, without posing questions of the justification, solvability, the intended aim, etc. All this has a negative impact on the final result when solving applied problems.

Why is it important to know how to solve applied problems? Because such problems form the majority, and the accumulation and standardization of the solution means result in a possibility to automate the solution in general. Besides, a correct understanding of the problem is a substantial and important part of the solution [1, 8].

The paper deals with questions of management of the applied problem solution.

2. Definition of an applied problem

We define first the notion of an “arbitrary problem”. For this we consider the basic components that are commonly used in its formulation and do not depend on the subject area, informal meaning of information, etc. (See Fig. 1).

The first element is the Cartesian product $I^{input} \times I^{output}$. The second element is the computational process $\Pr : I^{input} \to I^{output}$.

Thus, by a “problem” we mean a certain relation $T \subseteq I^{input} \times I^{output}$ for which at least two elements are defined $i_i^{in} \in I^{input}, i_k^{out} \in I^{output}$ ($i, k \in N$).

For stating an arbitrary problem T, it is necessary to explicitly specify elements $(i_i^{in}, i_k^{out}) \in T$, which define restrictions to the process $\Pr$.

Thus, by a “problem” we mean a certain relation $T \subseteq I^{input} \times I^{output}$ for which at least two elements are defined $i_i^{in} \in I^{input}, i_k^{out} \in I^{output}$ ($i, k \in N$).

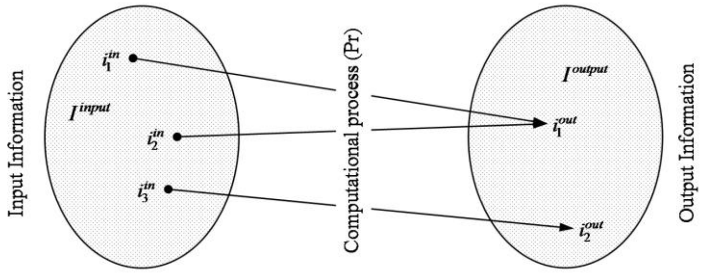

Fig. 1. The main components of a problem

Let T_0 denote a set of explicitly defined elements $(i_i^{in}, i_k^{out}) \in T$. It is easy to see that $T_0 \subseteq T$. In this case a “problem” can be defined through the relation $T \subseteq I^{input} \times I^{output}$ for which there is the implication: $T \Rightarrow T_0$.

Depending on the method of forming the Cartesian product $I^{input} \times I^{output}$, the class of problems T can be divided into two non-overlapping subsets [2, 4]. If the components I^{input}, I^{output} are associated with any real objects, then the corresponding problem will be called an “applied” one (T^{ap}). Otherwise, it is a theoretical problem (T^{th}).

We can certainly say that $T = T^{ap} \cup T^{th}$, but semantic relationship between these classes, apparently, does not exist.

As noted by several authors [1, 3, 4], applied problems are primary in relation to the theory, and therefore they are of particular interest. To establish a relationship between classes T^{ap} and T^{th}, it is necessary to determine the characteristic properties of each of them.

3. Formalization

In general, the computational process for the problem $T \in T^{ap}$ is unknown. Therefore, a model of the process is used for its solution.

It is reasonable to provide two levels [4] i.e. an informal level, at which the problem is formed, and a formal one, intended for building the model. Relationship between the levels and their specific content are universal. In the general scheme of problem solution the relationship is shown in Fig. 2.

At the formal level, problems are formed from the set T^{th}. Here, $T^{th} \subseteq X \times Y$, and $p_2 : X \to Y$. In this case the condition $T^{th} \neq \varnothing$ is ensured by the presence of the problem on the informal level with some additional conditions of consistency on coding (p_1) and interpretation (p_3) mapping.

Really, this means that a formal level for a new problem can be regarded as an informal one. Thus, the general scheme of problem solution is not limited to the single (primary) scheme shown in Fig.1. In fact, it can be considered as a superposition of the primary schemes. In this case the recursion [7] can be used for constructing a formal model:

$$\begin{aligned} &h((x,y),0) \cong f(i^{in}, i^{out}), \\ &h((x,y),i+1) \cong g((x,y),i,h((x,y),i)), \end{aligned} \tag{1}$$

where the two-place function $f(i^{in}, i^{out})$ is the process $\Pr(i^{in}) = i^{out}$, and $i \in N$. In regard to the three-place function g, its arguments are defined by the following relations

$$\begin{gathered} h((x,y),i) \cong h_{\tilde{k}_i}((x,y),i), \ \ (x,y) \in X_{\tilde{k}_i} \times Y_{\tilde{k}_i}, \\ \tilde{k}_i \cong \tilde{k}_{i-1} \times \tilde{N}, \tilde{k}_0 = \{(0)\}, \tilde{N} \subset N, |\tilde{N}| < \infty. \end{gathered}$$

Besides,

$$\begin{gathered} \forall i > 0 \, \exists p_1, p_2, p_3 \ p_1 : X_{\tilde{k}_{i-1}} \to X_{\tilde{k}_i}, p_2 : X_{\tilde{k}_i} \to Y_{\tilde{k}_i}, p_3 : Y_{\tilde{k}_i} \to Y_{\tilde{k}_{i-1}}, \\ h_{\tilde{k}_i}((x,y),i) \cong p_3 \circ p_2 \circ p_1(x). \end{gathered}$$

In this scheme there is only a unique informal level, at which the set T^{ap} is formed, and the final (in the limit - countable) set of formal levels, at which the corresponding sets T^{th} are formed.

Construction of a formal model with the use of the recursion (1), can be considered as a complete scheme of formalization necessary for solving any problem within T. It is necessary either to find a suitable scheme with the existing theoretical work, or somehow to build it.

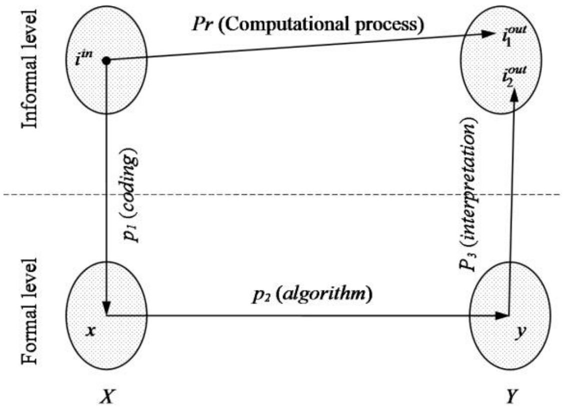

Fig. 2. Problem solution scheme

For any scheme there is a problem, which is thematically close to the problem of the algorithm statement [7]: how many levels is it necessary to build, and in what sense can we speak of the original problem solution? In terms of mathematical formalization, the latter question is directly related to the problem of justification [1, 5, 6].

4. Justification

Let's consider the problem solution scheme from T^{ap} in Fig. 2. We can write

$$\begin{aligned} &\Pr(i^{in}) = i_1^{out} \in I^{output}, \\ &p_3 \circ p_2 \circ p_1(i^{in}) = i_2^{out} \in I^{output}. \end{aligned} \qquad (2)$$

Realization of the condition (2) is associated with the computability [7] of the function (1). We call such a problem the algorithmically solvable one.

That is, we can state that for the problem there exists an algorithm that realizes the function (1) under all conditions associated with the implementation (1) and the fulfillment of the condition (2). But this is not enough, it is also necessary to establish a relationship between i_1^{out} and i_2^{out}. Ideally, it could take the form

$$\forall i^{in} \in I^{input} \ ((\Pr(i^{in}) = p_3 \circ p_2 \circ p_1(i^{in})) \Leftrightarrow (i_1^{out} = i_2^{out})) . \qquad (3)$$

Condition (3) makes sense on the entire set of the set T.

If we assume that we can formally prove the fulfillment of condition (3) on the set T, then the resulting solution we'll call the "justified" one. For example, justified are solutions of propositional calculus problems, but the propositional calculus this property no longer has [1, 5, 6]. These are the problems that arise in the process of problem construction of mathematical formalisms [1].

The condition (3) is stringent enough and for applied problems it is not appropriate [1, 4]. To be able to describe the nature of the solvability of problems with unjustified solutions we'll modify the condition.

Initially, we note that condition (3) is a consequence of one of the types of relationships, which can be introduced on the Cartesian product $I^{output} \times I^{output}$. The equivalence relation generated by the function of equality in this product. In general, any relationship $I^{out} \subseteq I^{output} \times I^{output}$ can be described by the function

$$\psi : I^{output} \times I^{output} \to R^{+},$$

where R^{+} is a subset of non-negative real numbers. In this case ψ realizes a variant of "similarity" of elements in I^{output}. This, for example, may be proximity ($\psi(i^{out}, i^{out}) = 0$), similarity ($\psi(i^{out}, i^{out}) = 1$) or some other variant.

Let's introduce a function of the type

$$\varphi : R^{+} \to [0,1],$$

and require that it be monotonic and satisfy the condition ($r, r_1, r_2 \in R^{+}$)

$$\varphi(r) = \begin{cases} 1, r = r_1, \\ 0, r = r_2. \end{cases} \qquad (4)$$

The selection of the function φ, corresponding to the condition (4), is determined by the nature of ψ mapping. In case of proximity $r_1 = 0, r_2 = +\infty$, and for similarity $r_1 = 1, r_2 = 0 \vee +\infty$. Obviously with such a choice, the superposition $\varphi \circ \psi$ makes sense. The superposition is the basis for introducing the function

$$\Phi(I^{input}) = \sum_{i^{in} \in I^{input}} \varphi(\psi(\Pr(i^{in}), p_3 \circ p_2 \circ p_1(i^{in}))) \cdot | I^{input} |^{-1}. \qquad (5)$$

It is easy to see that (3) is a special case of (5). With a suitable choice of φ and ψ mapping, the condition (3) can be written as: $\Phi(I^{input}) = 1$.

When calculating $\Phi(I^{input})$ there are only two possibilities

$$\begin{cases} \exists i^{in} \in I^{input} \ \forall \alpha \in [0,1] \ \Phi(i^{in}) \neq \alpha, & (6) \\ \forall i^{in} \in I^{input} \ \exists \alpha \in [0,1] \ \Phi(i^{in}) = \alpha. & (7) \end{cases}$$

Upon fulfillment of (6) Φ is noncomputable [7]. The reasons of noncomputability may be different. In the above formalization no restrictions are placed on the structure of sets I^{input}, I^{output}. And since they can be infinite, this can lead to a situation when resources are not sufficient to calculate Φ .The case (7) deals with the computability of Φ on the whole set I^{input} . And, specifically obtained number $\alpha \in [0,1]$ is in principle irrelevant.

We introduce two classes of problems. As stated above, upon $\Phi(I^{input}) = 1$ the solution of problem T is justified. It is known [1, 5], that the logical basis for this conclusion is the principle of deduction. And because the condition $\Phi(I^{input}) = 1$ is logically connected with computability at the whole interval $[0,1]$, then the principle of deduction can be extended to the whole class of problems for which there is the condition (7). Therefore, the relevant class of problems T can be called "deductively solvable" (or "algorithmically deductively solvable").

For the class of problems T with a noncomputable function Φ, in accordance with the problem definition, it is still possible to specify a subset T_0 : $\Phi(I_0^{input}) = 1$ (where I_0^{input} is the projection of the set I^{input} onto the subset T_0). Otherwise, the problem does not exist.

Between conditions $\Phi(I^{input}) = 1$ and $\Phi(I_0^{input}) = 1$ there is an obvious link

$$\Phi(I^{input}) = 1 \Rightarrow \Phi(I_0^{input}) = 1 .$$

However, the converse implication is interesting. Therefore, when trying to build something opposite to deductive solvability, it is appropriate to call this class of problems T "inductively solvable". Although, unlike the first case, no single principle of induction exists.

Is it possible to build a reverse implication? For example, in [9], such an implication is built for the so-called representative problem T_0 . However, this is done only for the problems of pattern recognition. But this result is easily generalized to the case of the problem of recognizing the truth. Other generalizations or similar results are unknown.

We now turn to the characterization of the set of problems T^{ap} . It contains problems of two classes: inductively solvable and insolvable ones (for which there is (6)). For the latter class there is always a possibility of transition to the inductively solvable class. This possibility appears as a result of cognition. And problems in the class of inductively solvable can never pass to the class of deductively solvable.

In turn, a set of problems T^{th} , can include problems of all three classes: deductively solvable, inductively solvable and insolvable. Moreover, the fundamental difference between T^{th} and T^{ap} is that for T^{th} and only for it, any problem can pass to a class of deductively solvable ones.

5. Methodological aspects

It was found that the characteristic features of a class of applied problems are:

- position in the solution scheme (only for such problems the informal level is directly connected with the reality);
- algorithmic inductive solvability/insolvability.

It is clear that the above features should influence the methods and way of organizing such a solution. These issues relate to the field of methodology [4], we consider them in more detail.

Let's make the corresponding particularization and specific filling of the solution shown in Fig. 2. At the informal level, there are several components that make up the solution process. Any problem is described by information, and obtaining a solution is associated with its processing. Therefore, the problem and the corresponding process have informational and computational components. Transformation of information is implemented in a certain environment, which naturally affects the informational and computational components. Therefore, we can single out another one, the so-called "technological" component.

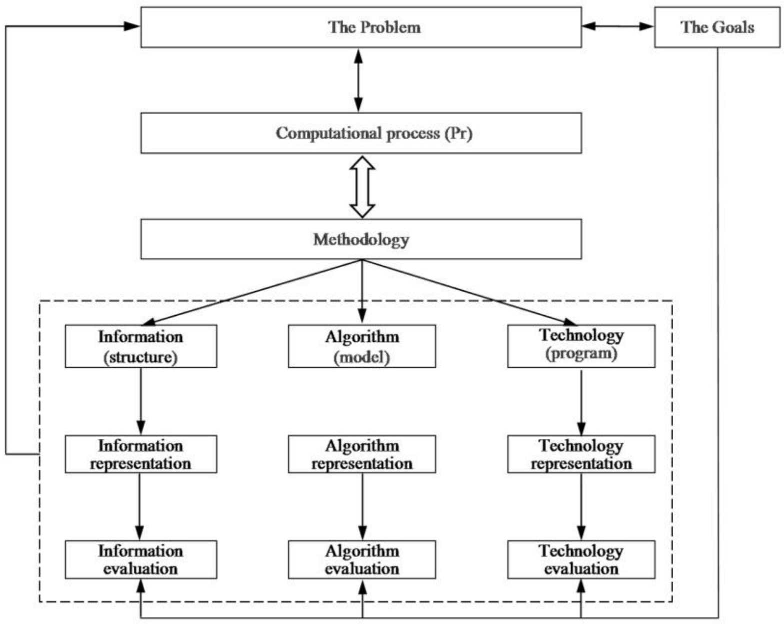

Fig. 3. Scheme of the general step of recursion

These components are naturally carried over to the level of formal constructions. The essence of management in this case is the solution of problems of representation and evaluation [6]. The problem of representation is typical for mathematical formalization. The concept of goal is associated with the two sides of the solution process: it is necessary to ensure the availability of a solution and to assess its quality from the standpoint of validity. These goals are also associated with the result and are characteristic of the process.

The implementation of any process is concerned with the recursion (1). At each step, a standard universal procedure is performed, the scheme of which is shown in Fig. 3. According to the results of evaluation, the problem can be corrected through its components. Among all the problems the one is chosen for which the best results are obtained. Further development of the theory occurs in the direction of such a problem as long as possible, or simply feasible.

The most general description of the methodology for problem solution is given above. It also applies to the inductively solvable problems from T^{th} .

The described approach to a practical problem does not pretend to any finality or completeness. Our aim is to show that investigating such a complex issue, it is possible to set some base points, but in the ideal case - the border.

References

1. M.Kline. *MATHEMATICS. The Loss of Certainty*. Oxford University Press, New York: 447 p. (1980)
2. V.Turchin. *The Phenomenon of Science*. Columbia University Press, New York: 368 p. (1977)
3. A.Foss. *The Essence of Mathematics*. Knizny Dom “Librokom”, Moscow:. 120 p. (2009)
4. Blekhman I.I., Myshkis A.D., Panovko Ya.G. *Mechanics and Applied Mathematics. Logics and Peculiarities of Mathematics Application*. Nauka, Moscow: 328 p. (1983)
5. V.Ya. Perminov. *Philosophy and Mathematics Justification*. Progress-Tradition, Moscow: 320 p. (2001)
6. Yu.A.Gastev. *Homomorphisms and Models. Logico-algebraic Aspects of Modeling*. Nauka, Moscow: 152 p. (1975)
7. N.Cutland. *Computability. An introduction to recursive function theory*. Cambridge University Press, London: 256 p. (1980)
8. G.Polya *How to Solve It: A New Aspect of Mathematical Method*. (2nd ed.), Princeton University Press, Princeton, New York: (1957)
9. V.V.Krasnoproshin, V.A. Obraztsov *Problem of Solvability and Choice of Algorithms for Decision Making by Precedence*. **Pattern Recognition and Image Analysis,** 16 (2): 155-169 (2006)