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ORDINARY DIFFERENTIAL EQUATIONS

# Existence of $\beta$-Weak Solutions of Stochastic Differential Equations with Measurable Right-Hand Sides

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In the present paper, we prove a theorem on the existence of $\beta$-weak solutions of the stochastic differential equation

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW(t), \qquad X \in R^d, \tag{1}$$

with Borel measurable functions $f : R_+ \times R^d \to R^d$ and $g : R_+ \times R^d \to R^{d\times d}$; here $W(t)$ is the $d$-dimensional Brownian motion. The existence theorem for weak solutions with Borel measurable locally bounded functions $f$ and $\sigma = gg^{\mathrm{T}}$ whose components satisfy Condition A below was obtained in [1]. In the present paper, we consider the case in which $f$ and $\sigma$ do not satisfy this condition. In this case, a weak solution of Eq. (1) is understood as a weak solution of some stochastic differential inclusion and is called a $\beta$-weak solution of Eq. (1). It was shown in [2, 3] that

• equation (1) has $\beta$-weak solutions provided that $f(t, X)$ and $g(t, X)$ are Borel measurable functions and have linear-order growth with respect to $X$ as $\|X\| \to \infty$;

• a $\beta$-weak solution of Eq. (1) is a weak solution of the stochastic inclusion

$$dX(t) \in \tilde{F}(t, X(t))dt + \tilde{G}(t, X(t))dW(t),$$

where $\tilde{F}(t, X)$ and $\tilde{G}(t, X)$ are the least convex closed sets containing all limit points of the functions $f(t, X')$ and $g(t, X')$ as $X' \to X$ on the set

$$\left\{(t, X) \mid \int\limits_{U(t,X)} \left(\det gg^{\mathrm{T}}(\tau, y)\right)^{-1} d\tau\, dy = \infty \quad \text{for each open neighborhood } U(t, X) \text{ of } (t, X)\right\}$$

of weak degeneracy of the mapping $g$ and consist of the unique points $f(t, X)$ and $g(t, X)$, respectively, on the set of weak nondegeneracy of $g$.

In the present paper, we consider differential inclusions with multimappings $F(t, X)$ and $G(t, X)$ such that, in general, $F \subset \tilde{F}$ and $G \subset \tilde{G}$. We prove the existence theorem for $\beta$-weak solutions of Eq. (1) assuming only that the functions $f(t, X)$ and $g(t, X)$ are locally bounded and Borel measurable.

Throughout the following, we use the notation in [1].

The matrix $\sigma(t, X) = g(t, X)g^{\mathrm{T}}(t, X)$ is symmetric and nonnegative. There exists a Borel measurable orthogonal matrix $T$ and a Borel measurable diagonal matrix $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_d)$ such that $\sigma = T\Lambda T^{\mathrm{T}}$. Let $g^* = T \operatorname{diag}\left(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_d}\,\right)$. Without loss of generality, we assume that $g = g^*$ in system (1) [4, pp. 97–98 of the Russian translation].

Take rows of the matrix $g$ with indices $\beta_1, \ldots, \beta_l$. Let

$$\sigma_{\beta_1,\ldots,\beta_l}(t, x_1, \ldots, x_d) = \operatorname{col}(g_{\beta_1}, \ldots, g_{\beta_l})\left(g_{\beta_1}^{\mathrm{T}} \cdots g_{\beta_l}^{\mathrm{T}}\right),$$

where $g_{\beta_j}$ is the $\beta_j$th row of $g$, and

$$D(0,a)=\left\{\left(x_{\beta_{l+1}},\ldots,x_{\beta_d}\right)\ \mid\ \left(x^2_{\beta_{l+1}}+\cdots+x^2_{\beta_d}\right)^{1/2}\le a\right\}.$$

Let us construct the set

$$H\left(\beta_1,\ldots,\beta_l\right)=\Big\{\left(t,x_{\beta_1},\ldots,x_{\beta_l}\right)\ \mid$$

for each open neighborhood[1] $U\left(t,x_{\beta_1},\ldots,x_{\beta_l}\right)$ of the point $\left(t,x_{\beta_1},\ldots,x_{\beta_l}\right)$, there exists a number $a>0$ such that the integral

$$\int\limits_{U(t,x_{\beta_1},\ldots,x_{\beta_l})}\ \sup_{(x_{\beta_{l+1}},\ldots,x_{\beta_d})\in D_2(0,a)}\left(\det\sigma_{\beta_1,\ldots,\beta_l}\left(t,x_1,\ldots,x_d\right)\right)^{-1}dt\,dx_{\beta_1}\ldots dx_{\beta_l}$$

is either undefined or equal to $\infty\Big\}$.

We say that a real function $h(t,X)=h\left(t,x_1,\ldots,x_d\right)$ satisfies **Condition A** if there exist rows $g_{\beta_1},\ldots,g_{\beta_l}$ of the matrix $g$ such that the function $h$ with fixed $\left(t,x_{\beta_1},\ldots,x_{\beta_l}\right)$ is continuous with respect to the remaining components $\left(x_{\beta_{l+1}},\ldots,x_{\beta_d}\right)$ of the vector $X$ and the set

$$\Big\{\left(t,x_1,\ldots,x_d\right)\mid\ \left(t,x_{\beta_1},\ldots,x_{\beta_l}\right)\in H\left(\beta_1,\ldots,\beta_l\right)\Big\}$$

lies in the set of points of continuity of the mapping $h$.

Consider the matrix function $\psi=\left(\psi^{ij}\left(t,x_1,\ldots,x_d\right)\right)$, $i=1,\ldots,d$, $j=1,\ldots,r$, and construct a multimapping $\Psi_0(t,X)$ by the following rule $(L)$. We split the set of all indices $\{(i,j)\mid i=1,\ldots,d,\ j=1,\ldots,r\}$ of components of the function $\psi$ into disjoint subsets $I^1_\psi,\ldots,I^n_\psi$ as follows: the indices $(i_1,j_1)$ and $(i_2,j_2)$ belong to a same subset only if the functions $\psi^{i_1j_1}$ and $\psi^{i_2j_2}$ are continuous with respect to same components of the vector $X$. If the components of the function $\psi$ with indices in $I^j_\psi$, $j\in\{1,\ldots,n\}$, are continuous with respect to the variables $\left(x_{\alpha^j_{m_j+1}},\ldots,x_{\alpha^j_d}\right)$ for each fixed $\left(t,x_{\alpha^j_1},\ldots,x_{\alpha^j_{m_j}}\right)$, we choose the rows of the matrix $g$ with indices $\beta^j_1,\ldots,\beta^j_{l_j}$ so as to ensure that the set $\{\beta^j_1,\ldots,\beta^j_{l_j}\}$ contains $\{\alpha^j_1,\ldots,\alpha^j_{m_j}\}$. Let us find the set $H\left(\beta^j_1,\ldots,\beta^j_{l_j}\right)$ and construct the $d\times r$ matrix $\psi_j$ with entries

$$\psi^{i_1i_2}_j=\begin{cases}\psi^{i_1i_2} & \text{if}\quad (i_1,i_2)\in I^j_\psi\\ 0 & \text{if}\quad (i_1,i_2)\notin I^j_\psi.\end{cases}$$

Let $\Psi_j(t,X)$ be the least convex closed set containing the matrix $\psi_j(t,X)$ and all of its limit points $\psi_j\left(t,X'\right)$ as $X'\to X$. We construct the multimappings $\Psi^0_j:R_+\times R^d\to\operatorname{cl}\left(R^{d\times r}\right)$ and $\Psi_0:R_+\times R^d\to\operatorname{cl}\left(R^{d\times r}\right)$ [$\operatorname{cl}(A)$ is the set of all nonempty closed subsets of a set $A$] as follows:

$$\Psi^0_j(t,X)=\begin{cases}\psi_j(t,X) & \text{if}\quad \left(t,x_{\beta^j_1},\ldots,x_{\beta^j_{l_j}}\right)\notin H^c\left(\beta^j_1,\ldots,\beta^j_{l_j}\right)\\ \Psi_j(t,X) & \text{if}\quad \left(t,x_{\beta^j_1},\ldots,x_{\beta^j_{l_j}}\right)\in H\left(\beta^j_1,\ldots,\beta^j_{l_j}\right),\end{cases}$$

$$\Psi_0=\Psi^0_1+\Psi^0_2+\cdots+\Psi^0_n$$

(the sum of a set $A\subset R$ and zero is by convention the set $A$ itself).

By the above-mentioned rule $(L)$, for the functions $f(t,X)$ and $\sigma(t,X)$, we construct the mappings $f_i(t,X)$, $F_i(t,X)$, $F^0_i(t,X)$, $i=1,\ldots,n_1$, $F_0(t,X)$ and $\sigma_j(t,X)$, $A_j(t,X)$, $A^0_j(t,X)$, $j=1,\ldots,n_2$, $A_0(t,X)$, respectively, where $n_1$ and $n_2$ are the numbers of subsets into which the index sets of the components of the functions $f(t,X)$ and $\sigma(t,X)$, respectively, are split.

[1] An open neighborhood is defined as a neighborhood open in the space of the variables $\left(t,x_{\beta_1},\ldots,x_{\beta_l}\right)$.

Note that for any choice of the rows of the matrix $g$, the set $\Psi_j^0(t,X)$ consists of the single matrix $\psi_j(t,X)$ provided that the components of the function $\psi(t,X)$ in the set $I_\psi^j$ are continuous with respect to $X$ for each fixed $t \in R_+$.

**Definition.** Suppose that there exists a process $X(t)$ defined on some probability space $(\Omega, \mathscr{F}, P)$ with the flow $\mathscr{F}_t$ of $\sigma$-algebras and satisfying the following conditions.

1. There exists an $(\mathscr{F}_t)$-stopping time $e$ such that the process $X(t)1_{[0,e)}(t)$ is $(\mathscr{F}_t)$-coordinated, has continuous trajectories for $t < e$ almost surely, and satisfies the condition $\limsup_{t\uparrow e} \|X(t)\| = \infty$ if $e < \infty$.

2. There exists an $(\mathscr{F}_t)$-Brownian motion $W(t)$ with $W(0) = 0$ almost surely.

3. There exist processes $v(t)$ and $u(t)$ defined on $(\Omega, \mathscr{F}, P)$ and such that $v(t)1_{[0,e)}(t)$ and $u(t)1_{[0,e)}(t)$ are measurable and $(\mathscr{F}_t)$-coordinated, the inclusions

$$v(t)1_{[0,e)}(t) \in F_0(t, X(t,\omega))1_{[0,e)}(t), \qquad u(t)u^{\mathrm{T}}(t)1_{[0,e)}(t) \in A_0(t, X(t,\omega))1_{[0,e)}(t)$$

hold for $(\mu \times P)$-almost all $(t,\omega) \in R_+ \times \Omega$, and $v \in L_1^{\mathrm{loc}}$ and $u \in L_2^{\mathrm{loc}}$.

4. The relation

$$X(t) = X(0) + \int_0^t v(\tau)d\tau + \int_0^t u(\tau)dW(\tau)$$

is valid with probability 1 for all $t \in [0, e)$.

Then the tuple $(\Omega, \mathscr{F}, P, \mathscr{F}_t, W(t), X(t), v(t), u(t), e)$ [or, briefly, $X(t)$] is referred to as a *$\beta$-weak solution* of Eq. (1).

A function $h : R_+ \times R^d \to R^{d\times r}$ is said to be *locally bounded* if, for each $b > 0$, there exists a constant $N(b)$ such that $\|h(t,X)\| \le N(b)$ for all $t \in [0,b]$ and $X \in B(0,b)$.

**Theorem.** *Let $f$ and $g$ be Borel measurable locally bounded functions. Then for any given probability $\nu$ on $\left(R^d, \mathscr{B}\left(R^d\right)\right)$, Eq.* (1) *has a $\beta$-weak solution with the initial distribution $\nu$.*

**Proof.** Let us construct the matrices $\bar\sigma_n = T\Lambda_n T^{\mathrm{T}}$, where

$$\begin{aligned}
\Lambda_n &= \operatorname{diag}\left((\lambda_1 + 1/n) \wedge n, \ldots, (\lambda_d + 1/n) \wedge n\right), \\
\bar g_n &= T \operatorname{diag}\left(((\lambda_1 + 1/n) \wedge n)^{1/2}, \ldots, ((\lambda_d + 1/n) \wedge n)^{1/2}\right), \\
\bar f_n(t,X) &= \left(f_n^i(t,X)\right), \qquad \bar f_n^i(t,X) = \left(f^i(t,X) \vee (-n)\right) \wedge n, \qquad i = 1,\ldots,d, \qquad n \in N.
\end{aligned}$$

For each positive integer $n$, there exists a constant $\alpha_n > 0$ such that $\det \bar g_n \bar g_n^{\mathrm{T}} = \det \bar\sigma_n \ge \alpha_n$ for all $(t,X) \in R_+ \times R^d$; in addition, $\lim_{n\to\infty} \bar f_n(t,X) = f(t,X)$ and $\lim_{n\to\infty} \bar\sigma_n(t,X) = \sigma(t,X)$ at each point $(t,X) \in R_+ \times R^d$.

By the Krylov theorem (Theorem II.6.1 in [5]), for each $n \in N$, the equation

$$X_n(t) = X_n(0) + \int_0^t \bar f_n\left(\tau, X_n(\tau)\right)d\tau + \int_0^t \bar g_n\left(\tau, X_n(\tau)\right)dW_n(\tau), \qquad t \in R_+, \tag{2}$$

has a weak solution $(\Omega_n, \mathscr{F}_n, P_n, \mathscr{F}_{nt}, W_n(t), X_n(t),\ t \in R_+)$ with the initial distribution $\nu$.

We define $\tau_n^m = \inf\{t \mid \|X_n(t)\| > m\}$ and $X_n^m(t) = X_n\left(t \wedge \tau_n^m\right)$ and consider the double sequence $\left(X_i^j, \tau_i^j\right)_{i,j=1}^\infty$.

Let

$$\Psi_k = \left(\left(X_k^1, \tau_k^1\right), \left(X_k^2, \tau_k^2\right), \ldots, \left(X_k^m, \tau_k^m\right), \ldots\right), \qquad k = 1, 2, \ldots$$

We introduce a metric $\varrho$ in $\left(C\left([0,+\infty), R^d\right), [0,+\infty]\right)$ and a metric $D$ in

$$\left(\left(C\left([0,+\infty), R^d\right), [0,+\infty]\right) \times \cdots \times \left(C\left([0,+\infty), R^d\right), [0,+\infty]\right) \times \cdots\right)$$

as follows:

$$\varrho\left((z,\tau),(z^1,\tau^1)\right)=\sum_{n=1}^{\infty}\frac{1}{2^n}\left(\sup_{0\le t\le n}\left\|z(t)-z^1(t)\right\|\wedge 1\right)+\left|\frac{\tau}{1+\tau}-\frac{\tau^1}{1+\tau^1}\right|,$$

$$D\left(\left(\left(X_n^1,\tau_n^1\right)\cdots\left(X_n^m,\tau_n^m\right)\cdots\right),\left(\left(X_k^1,\tau_k^1\right)\cdots\left(X_k^m,\tau_k^m\right)\cdots\right)\right)$$
$$=\sum_{m=1}^{\infty}\frac{1}{2^{m+1}}\varrho\left(\left(X_n^m,\tau_n^m\right),\left(X_k^m,\tau_k^m\right)\right).$$

The sequence $P^{\Psi_n}$, $n\ge 1$, is dense in the space

$$\left(\left(C\left([0,+\infty),R^d\right),[0,+\infty]\right)\times\cdots\times\left(C\left([0,+\infty),R^d\right),[0,+\infty]\right)\times\cdots\right)$$

[1, Lemma 3].

The sequence $\Psi_n$, $n\ge 1$, satisfies the assumptions of the Skorokhod theorem (see Theorem I.2.7 in [4]). It follows from the proof of Theorem I.2.7 in [4] that there exists a subsequence $n_k$ of the sequence $n$ (to simplify the notation, we write $n$ instead of $n_k$) and processes $\varepsilon_n=\left(\left(z_n^1,\eta_n^1\right),\ldots,\left(z_n^m,\eta_n^m\right),\ldots\right)$ and $\varepsilon=\left(\left(z^1,\eta^1\right),\ldots,\left(z^m,\eta^m\right),\ldots\right)$ on some probability space $(\Omega,\mathscr{F},P)$ such that $z_n^m(t)$ and $z^m(t)$ are continuous processes, $P^{\varepsilon_n}=P^{\Psi_n}$, $z_n^m(t)\to_{n\to\infty}z^m(t)$ uniformly on each compact set in $R_+$ almost surely, and $\eta_n^m\to_{n\to\infty}\eta^m$ almost surely. In addition, $z^m(t)=z^{m+1}(t)$ for $t<\eta^m$, and $\eta^m\le\eta^{m+1}$ almost surely. Let $e=\lim_{m\to\infty}\eta^m$. We define a process $z(t)$ as follows: $z(t)=z^m(t)$ for $t\le\eta^m$ if $\eta^m<\infty$, $z(t)=z^m(t)$ for $t<\eta^m$ if $\eta^m=\infty$, and $z(t)=0$ for $t\ge e$. By $\sigma_{t+\epsilon}$ we denote the least $\sigma$-algebra with respect to which all random processes $z^m(s)$, $0\le s\le t+\epsilon$, $m\ge 1$, are measurable. Let $\mathscr{F}_t=\bigcap_{\epsilon>0}\sigma_{t+\epsilon}$; then $z(t)1_{[0,e)}(t)$ is a $(\mathscr{F}_t)$-coordinated process and has continuous trajectories for $t<e$. In addition, $e$ is an $(\mathscr{F}_t)$-stopping time, and $\limsup_{t\uparrow e}\|z(t)\|=\infty$ for $e<\infty$.

For each of the sets $I_f^i$, $i\in\{1,\ldots,n_1\}$, and $I_\sigma^j$, $j\in\{1,\ldots,n_2\}$, we construct the vector $\left(\bar f_n\right)_i$ with components

$$\left(\bar f_n\right)_i^k=\begin{cases}\bar f_n^k & \text{for}\quad k\in I_f^i\\ 0 & \text{for}\quad k\notin I_f^i\end{cases}$$

and the matrix $\left(\bar\sigma_n\right)_j$ with entries

$$\left(\bar\sigma_n\right)_j^{i_1i_2}=\begin{cases}\bar\sigma_n^{i_1i_2} & \text{for}\quad (i_1,i_2)\in I_\sigma^j\\ 0 & \text{for}\quad (i_1,i_2)\notin I_\sigma^j.\end{cases}$$

For any $i\in\{1,\ldots,n_1\}$, $j\in\{1,\ldots,n_2\}$, $m\in N$, and $q\in N$, the sequences

$$\left(\bar f_n\right)_i\left(t,z_n^m(t)\right),\qquad\left(\bar\sigma_n\right)_j\left(t,z_n^m(t)\right),\qquad n\ge 1,$$

are relatively weakly compact in $L_1\left([0,q]\times\Omega,R^d\right)$ and $L_1\left([0,q]\times\Omega,R^{d\times d}\right)$, respectively. There exist subsequences $\left(\bar f_{n(1)}\right)_i\left(t,z_{n(1)}^1(t,\omega)\right)$ and $\left(\bar\sigma_{n(1)}\right)_j\left(t,z_{n(1)}^1(t,\omega)\right)$ converging weakly to $v_i^{(1)}(t,\omega)$ and $b_j^{(1)}(t,\omega)$, respectively, on $[0,\eta^1\wedge 1)\times\Omega$. Let $\left(\bar f_{n(2)}\right)_i\left(t,z_{n(2)}^2(t,\omega)\right)$ and $\left(\bar\sigma_{n(2)}\right)_j\left(t,z_{n(2)}^2(t,\omega)\right)$ be subsequences of $\left(\bar f_{n(1)}\right)_i\left(t,z_{n(1)}^2(t,\omega)\right)$ and $\left(\bar\sigma_{n(1)}\right)_j\left(t,z_{n(1)}^2(t,\omega)\right)$ weakly converging to $v_i^{(2)}(t,\omega)$ and $b_j^{(2)}(t,\omega)$, respectively, on $[\eta^1\wedge 1,\eta^2\wedge 2)\times\Omega$, and so on. Thus, we construct processes $v_i(t,\omega)$ and $b_j(t,\omega)$ such that $v_i(t,\omega)=v_i^{(m)}(t,\omega)$ and $b_j(t,\omega)=b_j^{(m)}(t,\omega)$ for

$$(t,\omega)\in\left[\eta^{m-1}\wedge(m-1),\eta^m\wedge m\right)\times\Omega,\qquad m=1,2,\ldots$$

(we assume that $\eta^0=0$); we set $v_i^k(t,\omega)=0$, $k=1,\ldots,d$, and $b_j^{i_1i_2}(t,\omega)=0$, $i_1,i_2=1,\ldots,d$, for $(t,\omega)\in[e,+\infty)\times\Omega$. Let

$$v(t,\omega)=v_1(t,\omega)+\cdots+v_{n_1}(t,\omega),\qquad b(t,\omega)=b_1(t,\omega)+\cdots+b_{n_2}(t,\omega).$$

For each $b > 0$, there exists a sequence $\delta_n(b) \downarrow 0$ as $n \to \infty$ such that $(\bar\sigma_n)_j(t,X) \in [A_j(t,X)]_{\delta_n}$ and $(\bar f_n)_i(t,X) \in [F_i(t,X)]_{\delta_n}$ for arbitrary $t \in [0,b]$ and $X \in B(0,b)$, $i = 1,\dots,n_1$, $j = 1,\dots,n_2$, where $[A]_\epsilon$ is the $\epsilon$-neighborhood of a set $A$. We have

$$v_i^{(m)}(t,\omega) \in \bigcap_{n=1}^{\infty} \overline{\text{co}} \bigcup_{k=n}^{\infty} (\bar f_k)_i (t, z_k^m(t,\omega)) \subset \bigcap_{n=1}^{\infty} \overline{\text{co}} \bigcup_{k=n}^{\infty} [F_i(t, z_k^m(t,\omega))]_{\delta_k(m)},$$

$$b_j^{(m)}(t,\omega) \in \bigcap_{n=1}^{\infty} \overline{\text{co}} \bigcup_{k=n}^{\infty} (\bar\sigma_k)_j (t, z_k^m(t,\omega)) \subset \bigcap_{n=1}^{\infty} \overline{\text{co}} \bigcup_{k=n}^{\infty} [A_j(t, z_k^m(t,\omega))]_{\delta_k(m)}$$

for $(\mu\times P)$-almost all $(t,\omega) \in [\eta^{m-1}\wedge(m-1), \eta^m\wedge m)\times\Omega$, $\delta_k(m) \downarrow 0$ as $k \to \infty$, where $\overline{\text{co}}(A)$ is the closed convex hull of a set $A$.

One can readily see that the mappings $F_i$ and $A_j$ are upper semicontinuous with respect to $X \in R^d$. Consequently, $v_i(t,\omega) \in F_i(t, z^m(t,\omega))$ and $b_j(t,\omega) \in A_j(t, z^m(t,\omega))$ for $(\mu\times P)$-almost all $(t,\omega) \in [\eta^{m-1}\wedge(m-1), \eta^m\wedge m)\times\Omega$. Let $\hat v_i(t,\omega)$ be the conditional expectation $E(v_i(t,\omega) \mid \mathscr{F}_t)$, let $\hat b_j(t,\omega)$ be the conditional expectation $E(b_j(t,\omega) \mid \mathscr{F}_t)$, let $\hat v(t,\omega) = \hat v_1(t,\omega)+\dots+\hat v_{n_1}(t,\omega)$, and let $\hat b(t,\omega) = \hat b_1(t,\omega)+\dots+\hat b_{n_2}(t,\omega)$. Then $\hat v_i(t,\omega) \in F_i(t, z^m(t,\omega))$ and $\hat b_j(t,\omega) \in A_j(t, z^m(t,\omega))$ for $(\mu\times P)$-almost all $(t,\omega) \in [\eta^{m-1}\wedge(m-1), \eta^m\wedge m)\times\Omega$. Let

$$B_m(I_f^i) = \Big\{(t,\omega) \in [\eta^{m-1}\wedge(m-1), \eta^m\wedge m)\times\Omega \;\Big|\; \Big(t, z^m_{\beta_1^i}(t,\omega),\dots, z^m_{\beta^i_{l_i}}(t,\omega)\Big) \in H(\beta_1^i,\dots,\beta^i_{l_i})\Big\},$$

$$B_m(I_\sigma^j) = \Big\{(t,\omega) \in [\eta^{m-1}\wedge(m-1), \eta^m\wedge m)\times\Omega \;\Big|\; \Big(t, z^m_{\beta_1^j}(t,\omega),\dots, z^m_{\beta^j_{l_j}}(t,\omega)\Big) \in H\Big(\beta_1^j,\dots,\beta^j_{l_j}\Big)\Big\},$$

$$B_m^c(I_f^i) = \big([\eta^{m-1}\wedge(m-1), \eta^m\wedge m)\times\Omega\big)\backslash B_m(I_f^i),$$

$$B_m^c(I_\sigma^j) = \big([\eta^{m-1}\wedge(m-1), \eta^m\wedge m)\times\Omega\big)\backslash B_m(I_\sigma^j).$$

We introduce the processes

$$\tilde v_i^{(m)}(t,\omega) = \begin{cases} f_i(t, z^m(t,\omega)) & \text{for} \quad (t,\omega) \in B_m^c(I_f^i) \\ \hat v_i(t,\omega) & \text{for} \quad (t,\omega) \in B_m(I_f^i), \end{cases}$$

$$\tilde b_j^{(m)}(t,\omega) = \begin{cases} \sigma_j(t, z^m(t,\omega)) & \text{for} \quad (t,\omega) \in B_m^c(I_\sigma^j) \\ \hat b_j(t,\omega) & \text{for} \quad (t,\omega) \in B_m(I_\sigma^j). \end{cases}$$

Let $\tilde v_i(t,\omega) = \tilde v_i^{(m)}(t,\omega)$ and $\tilde b_j(t,\omega) = \tilde b_j^{(m)}(t,\omega)$ for $(t,\omega) \in [\eta^{m-1}\wedge(m-1), \eta^m\wedge m)\times\Omega$. We set $\tilde v_i^k = 0$, $k = 1,\dots,d$, and $\tilde b_j^{i_1 i_2} = 0$, $i_1, i_2 = 1,\dots,d$, for $(t,\omega) \in [e,+\infty)\times\Omega$. Let

$$\tilde v(t,\omega) = \tilde v_1(t,\omega)+\tilde v_2(t,\omega)+\dots+\tilde v_{n_1}(t,\omega), \qquad \tilde b(t,\omega) = \tilde b_1(t,\omega)+\tilde b_2(t,\omega)+\dots+\tilde b_{n_2}(t,\omega).$$

For $(\mu\times P)$-almost all $(t,\omega) \in R_+\times\Omega$, the inclusions

$$\tilde v(t,\omega)1_{[0,e)}(t) \in F_0(t, z(t,\omega))1_{[0,e)}(t), \qquad \tilde b(t,\omega)1_{[0,e)}(t) \in A_0(t, z(t,\omega))1_{[0,e)}(t)$$

are valid, and $\tilde v(t,\omega)$ and $\tilde b(t,\omega)$ are $(\mathscr{F}_t)$-coordinated processes.

We fix $m \in N$ and choose arbitrary $s, t \in R_+$, $s \le t \le m$, an arbitrary twice continuously differentiable function $h : R^d \to R$ bounded together with its partial derivatives of order $\le 2$, and an arbitrary bounded continuous $\big(\mathscr{B}_s\big(C(R_+, R^d)\big)\big)$-measurable function $q : C(R_+, R^d) \to R$.

By using the Itô formula, from (2), we obtain

$$E_n\left(\left(\left(h\left(X_n^m(t)\right)-h\left(X_n^m(s)\right)-\int\limits_{s\wedge\tau_n^m}^{t\wedge\tau_n^m}\left(\frac{1}{2}\sum_{i,j=1}^{d}\bar{\sigma}_n^{ij}\left(\tau,X_n^m(\tau)\right)h_{x_ix_j}\left(X_n^m(\tau)\right)\right.\right.\right.\right.$$
$$\left.\left.\left.\left.+\sum_{i=1}^{d}\bar{f}_n^i\left(\tau,X_n^m(\tau)\right)h_{x_i}\left(X_n^m(\tau)\right)\right)d\tau\right)q\left(X_n^m\right)\right)=0,\qquad h_{x_i}=\frac{\partial h}{\partial x_i}. \tag{3}$$

We fix the component $f^i(t,X)$ of the vector $f$ with index $i$. By rule $(L)$, the function $f^i(t,X)$ is continuous with respect to the variables $\left(x_{\beta_{l+1}},\ldots,x_{\beta_d}\right)$ for each fixed $(t,x_{\beta_1},\ldots,x_{\beta_l})$; we set $(t,x_{\beta_1},\ldots,x_{\beta_l})=(t,\hat{x})$ and $(x_{\beta_l+1},\ldots,x_{\beta_d})=\hat{\hat{x}}$. (Without loss of generality, one can assume that $\beta_1=1,\ \ldots,\ \beta_l=l$.) Each of the processes $X_n$, $X_n^m$, $z$, $z_n^m$, and $z^m$ splits into two processes, $X_n=\left(\hat{X}_n,\hat{\hat{X}}_n\right)$, $X_n^m=\left(\hat{X}_n^m,\hat{\hat{X}}{}_n^m\right)$, $z=\left(\hat{z},\hat{\hat{z}}\right)$, $z_n^m=\left(\hat{z}_n^m,\hat{\hat{z}}{}_n^m\right)$, and $z^m=\left(\hat{z}^m,\hat{\hat{z}}{}^m\right)$. To simplify the notation, we write $H$ instead of $H\left(\beta_1,\ldots,\beta_l\right)$ and set $\left(\bar{\sigma}_n\right)_{1,\ldots,l}\left(t,x_1,\ldots,x_d\right)=a_n\left(t,\hat{x},\hat{\hat{x}}\right)$ and $\sigma_{1,\ldots,l}\left(t,x_1,\ldots,x_d\right)=a\left(t,\hat{x},\hat{\hat{x}}\right)$. Let

$$\begin{aligned}
&B_1(0,a)=\left\{\left(x_1,\ldots,x_l\right)\ \middle|\ \left(x_1^2+\cdots+x_l^2\right)^{1/2}\le a\right\},\qquad H^c=\left(R_+\times R^l\right)\backslash H,\\
&(H)_\gamma=\left\{(t,x)\in R_+\times R^l\ \middle|\ \sup_{(s,y)\in H}\left(|t-s|+\|x-y\|\right)<\gamma\right\},\\
&(H)_\gamma^c=\left(R_+\times R^l\right)\backslash(H)_\gamma.
\end{aligned}$$

Take a sequence $\epsilon_k\downarrow 0$ as $k\to\infty$.

By [1],

$$J_1\equiv\lim_{n\to\infty}E\left(\left(\int\limits_{s\wedge\eta_n^m}^{t\wedge\eta_n^m}1_{(H)^c_{\epsilon_k}}\left(\tau,\hat{z}_n^m(\tau)\right)f^i\left(\tau,\hat{z}_n^m(\tau),\hat{\hat{z}}{}_n^m(\tau)\right)h_{x_i}\left(\hat{z}_n^m(\tau),\hat{\hat{z}}{}_n^m(\tau)\right)d\tau\right)q\left(\hat{z}_n^m,\hat{\hat{z}}{}_n^m\right)\right)$$
$$=E\left(\left(\int\limits_{s\wedge\eta^m}^{t\wedge\eta^m}1_{(H)^c_{\epsilon_k}}\left(\tau,\hat{z}^m(\tau)\right)f^i\left(\tau,\hat{z}^m(\tau),\hat{\hat{z}}{}^m(\tau)\right)h_{x_i}\left(\hat{z}^m(\tau),\hat{\hat{z}}{}^m(\tau)\right)d\tau\right)q\left(\hat{z}^m,\hat{\hat{z}}{}^m\right)\right). \tag{4}$$

Let us show that

$$J_1=E\left(\left(\int\limits_{s\wedge\eta^m}^{t\wedge\eta^m}1_{(H)^c_{\epsilon_k}}\left(\tau,\hat{z}^m(\tau)\right)v^i(\tau)h_{x_i}\left(\hat{z}^m(\tau),\hat{\hat{z}}{}^m(\tau)\right)d\tau\right)q\left(\hat{z}^m,\hat{\hat{z}}{}^m\right)\right). \tag{5}$$

For each positive integer $k$, we construct a sequence of continuous functions $\varphi_j:R_+\times R^l\to[0,1]$ such that $\varphi_j\le 1_{(H)^c_{\epsilon_k}}$ and $\varphi_j\uparrow 1_{(H)^c_{\epsilon_k}}$ as $j\to\infty$.

We have

$$\lim_{n\to\infty}E\left(\left(\int\limits_{s\wedge\eta_n^m}^{t\wedge\eta_n^m}\varphi_j\left(\tau,\hat{z}_n^m(\tau)\right)f^i\left(\tau,\hat{z}_n^m(\tau),\hat{\hat{z}}{}_n^m(\tau)\right)h_{x_i}\left(\hat{z}_n^m(\tau),\hat{\hat{z}}{}_n^m(\tau)\right)d\tau\right)q\left(\hat{z}_n^m,\hat{\hat{z}}{}_n^m\right)\right)$$
$$=\lim_{n\to\infty}E\left(\int\limits_{s\wedge\eta_n^m}^{t\wedge\eta_n^m}\left(\varphi_j\left(\tau,\hat{z}_n^m(\tau)\right)h_{x_i}\left(\hat{z}_n^m(\tau),\hat{\hat{z}}{}_n^m(\tau)\right)q\left(\hat{z}_n^m,\hat{\hat{z}}{}_n^m\right)\right.\right.$$

$$- \varphi_j(\tau, \hat{z}^m(\tau))\, h_{x_i}\left(\hat{z}^m(\tau), \hat{\hat{z}}^m(\tau)\right) q\left(\hat{z}^m, \hat{\hat{z}}^m\right)\Big) f^i\left(\tau, \hat{z}_n^m(\tau), \hat{\hat{z}}_n^m(\tau)\right) d\tau\Big)$$

$$+ \lim_{n\to\infty} E\Bigg(\int\limits_{s\wedge\eta^m}^{t\wedge\eta^m} \varphi_j(\tau, \hat{z}^m(\tau))\, h_{x_i}\left(\hat{z}^m(\tau), \hat{\hat{z}}^m(\tau)\right) q\left(\hat{z}^m, \hat{\hat{z}}^m\right)$$

$$\times \left(f^i\left(\tau, \hat{z}_n^m(\tau), \hat{\hat{z}}_n^m(\tau)\right) - \tilde{v}^i(\tau)\right) d\tau\Bigg) + \lim_{n\to\infty} E\Bigg(\int\limits_{t\wedge\eta^m}^{t\wedge\eta_n^m} \varphi_j(\tau, \hat{z}^m(\tau))$$

$$\times h_{x_i}\left(\hat{z}^m(\tau), \hat{\hat{z}}^m(\tau)\right) q\left(\hat{z}^m, \hat{\hat{z}}^m\right) f^i\left(\tau, \hat{z}_n^m(\tau), \hat{\hat{z}}_n^m(\tau)\right) d\tau\Bigg)$$

$$+ \lim_{n\to\infty} E\Bigg(\int\limits_{s\wedge\eta_n^m}^{s\wedge\eta^m} \varphi_j(\tau, \hat{z}^m(\tau))\, h_{x_i}\left(\hat{z}^m(\tau), \hat{\hat{z}}^m(\tau)\right) q\left(\hat{z}^m, \hat{\hat{z}}^m\right) f^i\left(\tau, \hat{z}_n^m(\tau), \hat{\hat{z}}_n^m(\tau)\right) d\tau\Bigg)$$

$$+ E\Bigg(\int\limits_{s\wedge\eta^m}^{t\wedge\eta^m} \varphi_j(\tau, \hat{z}^m(\tau))\, h_{x_i}\left(\hat{z}^m(\tau), \hat{\hat{z}}^m(\tau)\right) q\left(\hat{z}^m, \hat{\hat{z}}^m\right) v^i(\tau) d\tau\Bigg) \tag{6}$$

for each positive integer $j$. In (6), the first term on the right-hand side is zero, since

$$z_n^m(\tau) \underset{n\to\infty}{\longrightarrow} z^m(\tau)$$

uniformly with respect to $\tau \in [0,t]$ almost surely; the second term is zero by virtue of the weak convergence of $\bar{f}_n^i(\tau, z_n^m(\tau))$ to $v^i(\tau)$ in $L_1([0, t\wedge\eta^m)\times\Omega, R)$ (to simplify the notation, we assume that the sequence $\bar{f}_n^i(\tau, z_n^m(\tau))$ itself converges weakly to $v^i(\tau)$ in $L_1([0, t\wedge\eta^m)\times\Omega, R)$), the definition of $\bar{f}_n^i$, and the local boundedness of $f^i$; and the third and fourth term are zero, since $t\wedge\eta_n^m \to_{n\to\infty} t\wedge\eta^m$ and $s\wedge\eta_n^m \to_{n\to\infty} s\wedge\eta^m$ almost surely.

By using Corollary 1 in [1], we obtain the relations

$$\lim_{j\to\infty} \limsup_{n\to\infty} E\Bigg|\Bigg(\int\limits_{s\wedge\eta^m}^{t\wedge\eta^m} \left(1_{(H)^c_{\epsilon_k}}(\tau, \hat{z}_n^m(\tau)) - \varphi_j(\tau, \hat{z}_n^m(\tau))\right) f^i\left(\tau, \hat{z}_n^m(\tau), \hat{\hat{z}}_n^m(\tau)\right)$$

$$\times h_{x_i}\left(\hat{z}_n^m(\tau), \hat{\hat{z}}_n^m(\tau)\right) d\tau\Bigg) q\left(\hat{z}_n^m, \hat{\hat{z}}_n^m\right)\Bigg|$$

$$\le C_5 \lim_{j\to\infty} \limsup_{n\to\infty} E\Bigg(\int\limits_{s\wedge\eta^m}^{t\wedge\eta^m} 1_{(H)^c_{\epsilon_k}}(\tau, \hat{z}_n^m(\tau)) \left(1_{(H)^c_{\epsilon_k}}(\tau, \hat{z}_n^m(\tau)) - \varphi_j(\tau, \hat{z}_n^m(\tau))\right) d\tau\Bigg)$$

$$\le C_6 \lim_{j\to\infty} \Bigg(\int\limits_{([0,t]\times B_1(0,m))\cap(H)^c_{\epsilon_k}} \sup_{\|\hat{\hat{x}}\|\le m} \left(\left(\det a\left(\tau, \hat{x}, \hat{\hat{x}}\right)\right)\right)^{-1}$$

$$\times \left(1_{(H)^c_{\epsilon_k}}(\tau, \hat{x}) - \varphi_j(\tau, \hat{x})\right)^{l+1}\Bigg) d\tau\, d\hat{x}\Bigg)^{1/(l+1)} = 0; \tag{7}$$

in addition,

$$\lim_{j\to\infty} E\Bigg(\Bigg(\Bigg(\int\limits_{s\wedge\eta^m}^{t\wedge\eta^m} \left(1_{(H)^c_{\epsilon_k}}(\tau, \hat{z}^m(\tau)) - \varphi_j(\tau, \hat{z}^m(\tau))\right) v^i(\tau)$$

$$\times h_{x_i}\left(\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right)d\tau\Bigg)q\left(\hat{z}^m,\hat{\hat{z}}^m\right)\Bigg)=0. \tag{8}$$

Relations (6)–(8) imply the desired relation (5). Since $\bar{f}_n^i(\tau, z_n^m(\tau))$ weakly converges to $v^i(\tau)$ in $L_1([0, t\wedge\eta^m)\times\Omega, R)$, it follows from the definition of $\bar{f}_n^i$ and the local boundedness of $f^i$ that

$$\lim_{n\to\infty} E\left(\left(\int\limits_{s\wedge\eta_n^m}^{t\wedge\eta_n^m} f^i\left(\tau,\hat{z}_n^m(\tau),\hat{\hat{z}}_n^m(\tau)\right)h_{x_i}\left(\hat{z}_n^m(\tau),\hat{\hat{z}}_n^m(\tau)\right)d\tau\right)q\left(\hat{z}_n^m,\hat{\hat{z}}_n^m\right)\right)$$
$$=E\left(\left(\int\limits_{s\wedge\eta^m}^{t\wedge\eta^m} v^i(\tau)h_{x_i}\left(\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right)d\tau\right)q\left(\hat{z}^m,\hat{\hat{z}}^m\right)\right). \tag{9}$$

From relation (5), we find that there exists a sequence $k_n \underset{n\to\infty}{\to} +\infty$ such that

$$J_2 \equiv \lim_{n\to\infty} E\left(\left(\int\limits_{s\wedge\eta_n^m}^{t\wedge\eta_n^m} 1_{(H)^c_{\epsilon_{k_n}}}\left(\tau,\hat{z}_n^m(\tau)\right) f^i\left(\tau,\hat{z}_n^m(\tau),\hat{\hat{z}}_n^m(\tau)\right)h_{x_i}\left(\hat{z}_n^m(\tau),\hat{\hat{z}}_n^m(\tau)\right)d\tau\right)q\left(\hat{z}_n^m,\hat{\hat{z}}_n^m\right)\right)$$
$$=E\left(\left(\int\limits_{s\wedge\eta^m}^{t\wedge\eta^m} 1_{H^c}\left(\tau,\hat{z}^m(\tau)\right)v^i(\tau)h_{x_i}\left(\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right)d\tau\right)q\left(\hat{z}^m,\hat{\hat{z}}^m\right)\right). \tag{10}$$

It follows from (9) and (10) that

$$\lim_{n\to\infty} E\left(\left(\int\limits_{s\wedge\eta_n^m}^{t\wedge\eta_n^m} 1_{(H)_{\epsilon_{k_n}}}\left(\tau,\hat{z}_n^m(\tau)\right) f^i\left(\tau,\hat{z}_n^m(\tau),\hat{\hat{z}}_n^m(\tau)\right)h_{x_i}\left(\hat{z}_n^m(\tau),\hat{\hat{z}}_n^m(\tau)\right)d\tau\right)q\left(\hat{z}_n^m,\hat{\hat{z}}_n^m\right)\right)$$
$$=E\left(\left(\int\limits_{s\wedge\eta^m}^{t\wedge\eta^m} 1_{H}\left(\tau,\hat{z}^m(\tau)\right)v^i(\tau)h_{x_i}\left(\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right)d\tau\right)q\left(\hat{z}^m,\hat{\hat{z}}^m\right)\right). \tag{11}$$

By comparing (4) and (5), we obtain

$$E\left(\left(\int\limits_{s\wedge\eta^m}^{t\wedge\eta^m} 1_{(H)^c_{\epsilon_k}}\left(\tau,\hat{z}^m(\tau)\right) f^i\left(\tau,\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right)h_{x_i}\left(\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right)d\tau\right)q\left(\hat{z}^m,\hat{\hat{z}}^m\right)\right)$$
$$=E\left(\left(\int\limits_{s\wedge\eta^m}^{t\wedge\eta^m} 1_{(H)^c_{\epsilon_k}}\left(\tau,\hat{z}^m(\tau)\right)v^i(\tau)h_{x_i}\left(\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right)d\tau\right)q\left(\hat{z}^m,\hat{\hat{z}}^m\right)\right). \tag{12}$$

The comparison of (10) and (12) implies the relation

$$J_2=E\left(\left(\int\limits_{s\wedge\eta^m}^{t\wedge\eta^m} 1_{H^c}\left(\tau,\hat{z}^m(\tau)\right) f^i\left(\tau,\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right)h_{x_i}\left(\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right)d\tau\right)q\left(\hat{z}^m,\hat{\hat{z}}^m\right)\right). \tag{13}$$

By taking into account (11) and (13) and the relation $P^{\Psi_n}=P^{\varepsilon_n}$, we obtain

$$E\left(\left(\int\limits_{s\wedge\eta^m}^{t\wedge\eta^m} \tilde{v}^i(\tau)h_{x_i}\left(\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right)d\tau\right)q\left(\hat{z}^m,\hat{\hat{z}}^m\right)\right)$$

$$= E\left(\left(\int\limits_{s\wedge\eta^m}^{t\wedge\eta^m} 1_{H^c}\left(\tau,\hat{z}^m(\tau)\right) f^i\left(\tau,\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right) h_{x_i}\left(\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right) d\tau\right) q\left(\hat{z}^m,\hat{\hat{z}}^m\right)\right)$$

$$+ E\left(\left(\int\limits_{s\wedge\eta^m}^{t\wedge\eta^m} 1_{H}\left(\tau,\hat{z}^m(\tau)\right) v^i(\tau) h_{x_i}\left(\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right) d\tau\right) q\left(\hat{z}^m,\hat{\hat{z}}^m\right)\right)$$

$$= \lim_{n\to\infty} E\left(\left(\int\limits_{s\wedge\eta_n^m}^{t\wedge\eta_n^m} 1_{(H)^c_{\epsilon_{k_n}}}\left(\tau,\hat{z}_n^m(\tau)\right) f^i\left(\tau,\hat{z}_n^m(\tau),\hat{\hat{z}}_n^m(\tau)\right) h_{x_i}\left(\hat{z}_n^m(\tau),\hat{\hat{z}}_n^m(\tau)\right) d\tau\right.\right.$$

$$\left.\left.\times\, q\left(\hat{z}_n^m,\hat{\hat{z}}_n^m\right)\right) + \lim_{n\to\infty} E\left(\left(\int\limits_{s\wedge\eta_n^m}^{t\wedge\eta_n^m} 1_{(H)_{\epsilon_{k_n}}}\left(\tau,\hat{z}_n^m(\tau)\right) f^i\left(\tau,\hat{z}_n^m(\tau),\hat{\hat{z}}_n^m(\tau)\right)\right.\right.$$

$$\left.\left.\times\, h_{x_i}\left(\hat{z}_n^m(\tau),\hat{\hat{z}}_n^m(\tau)\right) d\tau\right) q\left(\hat{z}_n^m,\hat{\hat{z}}_n^m\right)\right)$$

$$= \lim_{n\to\infty} E\left(\left(\int\limits_{s\wedge\eta_n^m}^{t\wedge\eta_n^m} f^i\left(\tau,\hat{z}_n^m(\tau),\hat{\hat{z}}_n^m(\tau)\right) h_{x_i}\left(\hat{z}_n^m(\tau),\hat{\hat{z}}_n^m(\tau)\right) d\tau\right) q\left(\hat{z}_n^m,\hat{\hat{z}}_n^m\right)\right)$$

$$= \lim_{n\to\infty} E\left(\left(\int\limits_{s\wedge\eta_n^m}^{t\wedge\eta_n^m} \bar{f}_n^i\left(\tau,\hat{z}_n^m(\tau),\hat{\hat{z}}_n^m(\tau)\right) h_{x_i}\left(\hat{z}_n^m(\tau),\hat{\hat{z}}_n^m(\tau)\right) d\tau\right) q\left(\hat{z}_n^m,\hat{\hat{z}}_n^m\right)\right)$$

$$= \lim_{n\to\infty} E\left(\left(\int\limits_{s\wedge\tau_n^m}^{t\wedge\tau_n^m} \bar{f}_n^i\left(\tau,\hat{X}_n^m(\tau),\hat{\hat{X}}_n^m(\tau)\right) h_{x_i}\left(\hat{X}_n^m(\tau),\hat{\hat{X}}_n^m(\tau)\right) d\tau\right) q\left(\hat{X}_n^m,\hat{\hat{X}}_n^m\right)\right). \tag{14}$$

By using similar considerations, one can show that

$$\lim_{n\to\infty} E\left(\left(\int\limits_{s\wedge\tau_n^m}^{t\wedge\tau_n^m} \bar{\sigma}_n^{ij}\left(\tau,X_n^m(\tau)\right) h_{x_ix_j}\left(X_n^m(\tau)\right) d\tau\right) q\left(X_n^m\right)\right)$$
$$= E\left(\left(\int\limits_{s\wedge\eta^m}^{t\wedge\eta^m} \tilde{b}^{ij}(\tau) h_{x_ix_j}\left(z^m(\tau)\right) d\tau\right) q\left(z^m\right)\right) \tag{15}$$

for arbitrary fixed $i,j\in\{1,\ldots,d\}$.

It follows from (3), (14), and (15) that

$$E\left(\left(h\left(z^m(t)\right)-h\left(z^m(s)\right)-\int\limits_{s\wedge\eta^m}^{t\wedge\eta^m}\left(\frac{1}{2}\sum_{i,j=1}^d \tilde{b}^{ij}(\tau)h_{x_ix_j}\left(z^m(\tau)\right)\right.\right.\right.$$
$$\left.\left.\left.+\sum_{i=1}^d \tilde{v}^i(\tau)h_{x_i}\left(z^m(\tau)\right)\right)d\tau\right)q\left(z^m\right)\right)=0;$$

therefore, the process

$$h(z(t))-h(z(0))-\int\limits_0^t\left(\frac{1}{2}\sum_{i,j=1}^d \tilde{b}^{ij}(\tau)h_{x_ix_j}(z(\tau))+\sum_{i=1}^d \tilde{v}^i(\tau)h_{x_i}(z(\tau))\right)d\tau$$

is a local $(\mathscr{F}_t)$-martingale. The matrix $\tilde{b}(t,\omega)$ can be represented in the form

$$\tilde{b}(t,\omega) = Q(t,\omega)D(t,\omega)Q^{\mathrm{T}}(t,\omega),$$

where $Q(t,\omega)$ is an orthogonal matrix and $D(t,\omega)$ is a diagonal matrix with nonnegative entries; in addition, all entries of the matrices $Q(t,\omega)$ and $D(t,\omega)$ are measurable $(\mathscr{F}_t)$-coordinated processes. Let $\tilde{u}(t,\omega) = Q(t,\omega)\sqrt{D(t,\omega)}$; then

$$\tilde{u}(t,\omega)u^{\mathrm{T}}(t,\omega)1_{[0,e)}(t) \in A_0(t, z(t,\omega))1_{[0,e)}(t)$$

for $(\mu \times P)$-almost all $(t,\omega) \in R_+ \times \Omega$.

By [4, pp. 159–160 of the Russian translation], on the extension $\big(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P}\big)$ with the flow $\tilde{\mathscr{F}_t}$ of $\sigma$-algebras of the probability space $(\Omega, \mathscr{F}, P)$ with the flow $\mathscr{F}_t$ of $\sigma$-algebras, one can define a $\big(\tilde{\mathscr{F}_t}\big)$-Brownian motion $\tilde{W}(t)$ with $\tilde{W}(0) = 0$ almost surely such that

$$z(t) = z(0) + \int\limits_0^t \tilde{v}(\tau)d\tau + \int\limits_0^t \tilde{u}(\tau)d\tilde{W}(\tau)$$

with probability 1 for any $t \in [0,e)$.

Consequently, $\big(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P}, \tilde{\mathscr{F}_t}, \tilde{W}(t), z(t), \tilde{v}(t), \tilde{u}(t), e\big)$ is a $\beta$-weak solution of Eq. (1). The proof of the theorem is complete.

Consider the example

$$dx_1(t) = \big(r\,(x_1(t)) + tx_2^2(t)\big)\,dt + dW_1(t), \qquad dx_2(t) = -r\,(x_2(t))\,dt,$$

where $r(x) = \begin{cases} 1 & \text{if} \quad x \ge 0, \\ -1 & \text{if} \quad x < 0. \end{cases}$ The function $\sigma = gg^{\mathrm{T}}$ is continuous; therefore, $A_0\,(t, x_1, x_2)$ coincides with $\sigma\,(t, x_1, x_2) = \operatorname{diag}(1, 0)$. We split the set of indices of the components of the function $f$ into the subsets $I_f^1 = \{1\}$ and $I_f^2 = \{2\}$. For the function

$$f^1\,(t, x_1, x_2) = r\,(x_1) + tx_2^2,$$

we choose the first row of the matrix $g$; the set $H(1)$ is empty; therefore,

$$F_1^0\,(t, x_1, x_2) = \operatorname{col}\big(f^1\,(t, x_1, x_2)\,, 0\big)\,.$$

For $f^2\,(t, x_1, x_2) = -r\,(x_2)$, we choose the second row of the matrix $g$; we have

$$H(2) \times \{x_1 \in R\} = R_+ \times R^2,$$

and hence we obtain $F_2^0\,(t, x_1, x_2) = \operatorname{col}\,(0, -r\,(x_2))$ if $x_2 \ne 0$ and $F_2^0\,(t, x_1, 0) = (0, [-1, 1])^{\mathrm{T}}$. Therefore, $F_0\,(t, x_1, x_2) = \operatorname{col}\,(r\,(x_1) + tx_2^2, -r\,(x_2))$ if $x_2 \ne 0$ and $F_0\,(t, x_1, 0) = \operatorname{col}\,(r\,(x_1) + tx_2^2, [-1, 1])$. For the considered system, the results in [1] do not permit one to analyze the existence of weak solutions with an arbitrary initial distribution. The theorems in [2, 3] provide the existence of $\beta$-weak solutions with an arbitrary initial distribution, but for our example, the set $\tilde{F}\,(t, x_1, x_2)$ is substantially wider than the set $F_0\,(t, x_1, x_2)$.

**Remark.** Let us define the set

$$E = \Big\{(t, X) \in R_+ \times R^d \mid f(s, X) = 0,\ g(s, X) = 0 \text{ for almost all } s \ge t\Big\},$$

which we refer to as the set of zeros of the mappings $f$ and $g$. We say that a real-valued function $h(t, X) = h\,(t, x_1, \ldots, x_d)$ satisfies **Condition B** if there exist rows of the matrix $g$ with indices

$\beta_1, \ldots, \beta_l$ such that the function $h$ with fixed $(t, x_{\beta_1}, \ldots, x_{\beta_l})$ is continuous with respect to the remaining components $\left(x_{\beta_{l+1}}, \ldots, x_{\beta_d}\right)$ of the vector $X$ and the set

$$\{(t, x_1, \ldots, x_d) \mid (t, x_{\beta_1}, \ldots, x_{\beta_l}) \in H(\beta_1, \ldots, \beta_l)\}$$

lies in the union of the set of points of continuity of the function $h$ and the set of zeros of the mappings $f$ and $g$. One can readily see that if the components of the functions $f$ and $\sigma$ satisfy Condition B, then, for a $\beta$-weak solution of Eq. (1), one can construct a weak solution of Eq. (1). Therefore, the theorem proved above implies the following assertion.

**Corollary.** *Let $f(t, X)$ and $g(t, X)$ be Borel measurable locally bounded functions, and let the components of the functions $f(t, X)$ and $\sigma(t, X) = g(t, X)g^{\mathrm{T}}(t, X)$ satisfy Condition* B. *Then for any given probability $\nu$ on $\left(R^d, \mathscr{B}\left(R^d\right)\right)$, Eq.* (1) *has a weak solution with initial distribution $\nu$.*

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