Differential Equations, Vol. 37, No. 2, 2001, pp. 257–260. Translated from Differentsial'nye Uravneniya, Vol. 37, No. 2, 2001, pp. 238–241.


## PARTIAL DIFFERENTIAL EQUATIONS

# Solution of the Heat Equation with Mixed Boundary Conditions on the Surface of an Isotropic Half-Space

**P. A. Mandrik**

*Belarussian State University, Belarus*

Received November 16, 1999

Consider the nonstationary heat equation

$$T_{rr}(r,z,\tau)+r^{-1}T_r(r,z,\tau)+T_{zz}(r,z,\tau)=a^{-1}T_\tau(r,z,\tau) \tag{1}$$

in cylindrical coordinates ($r>0$, $z>0$, $\tau>0$) in the case of axial symmetry under the initial condition $T(r,z,0)=0$ and the mixed boundary conditions

$$-T_z(r,0,\tau)=\lambda^{-1}q_1(\tau)q_2(r), \qquad 0<r<R, \qquad \tau>0, \tag{2}$$

$$T(r,0,\tau)=0, \quad R<r<\infty, \qquad \tau>0. \tag{3}$$

The mathematical statement of this problem describes, in particular, a nonstationary temperature field under a local heating of an isotropic half-space (with temperature conductivity $a>0$ and thermal conductivity $\lambda>0$) through a circular domain of radius $R$ on the surface $z=0$; moreover, the surface of the half-space is thermally insulated outside this circular domain. In this case, obviously, the homogeneous initial condition and the boundary condition (3) do not affect the generality of the statement of the problem.

Applying the Laplace transform $\bar f(s)=\int_0^\infty f(\tau)\exp(-s\tau)dx$, $\operatorname{Re}s>0$, to problem (1)–(3), for the Laplace transform of the temperature we have

$$\bar T_{rr}(r,z,s)+r^{-1}\bar T_r(r,z,s)+\bar T_{zz}(r,z,s)=\sigma\bar T(r,z,s), \qquad \sigma=s/a, \tag{4}$$

$$-\bar T_z(r,0,s)=\lambda^{-1}\bar q_1(s)q_2(r), \qquad 0<r<R, \tag{5}$$

$$\bar T(r,0,s)=0, \qquad R<r<\infty. \tag{6}$$

From now on, for brevity, we omit the obvious constraint $\operatorname{Re}s>0$ on the parameter $s$.

Taking into account the boundedness of $\bar T(r,z,s)$ as $\sqrt{r^2+z^2}\to\infty$ and using separation of variables or the Hankel transform

$$H\left[\bar T(r,z,s)\right]=\bar T_H(p,z,s)=\int_0^\infty \bar T(r,z,s)J_0(pr)r\,dr, \qquad p>0,$$

we can write out the solution of Eq. (4) in the form

$$\bar T(r,z,s)=\int_0^\infty \bar C(p,s)\exp\left(-z\sqrt{p^2+\sigma}\right)J_0(pr)dp \tag{7}$$

and represent the normal derivative on the surface $z=0$ as

$$-\bar T_z(r,z,s)=\int_0^\infty \bar C(p,s)\sqrt{p^2+\sigma}\exp\left(-z\sqrt{p^2+\sigma}\right)J_0(pr)dp, \tag{8}$$

where $\bar{C}(p,s)$ is the unknown transform to be determined and $J_0(pr)$ is the Bessel function of the first kind and zero order.

Taking into account the mixed boundary conditions (5) and (6), for $\bar{C}(p,s)$, we obtain the pair of integral equations

$$\int_0^\infty \bar{C}(p,s)\sqrt{p^2+\sigma}\,J_0(pr)dp = \lambda^{-1}\bar{q}_1(s)q_2(r), \qquad 0<r<R, \tag{9}$$

$$\int_0^\infty \bar{C}(p,s)J_0(pr)dp = 0, \qquad R<r<\infty, \tag{10}$$

in the domain of $L$-transforms.

By choosing the functions $\bar{C}(p,s)$ in the form [1]

$$\bar{C}(p,s) = \frac{p}{\sqrt{p^2+\sigma}}\int_0^R \bar{\varphi}(t,s)\sin\left(t\sqrt{p^2+\sigma}\,\right)dt, \tag{11}$$

where $\bar{\varphi}(t,s)$ is some new unknown analytic function, we guarantee the validity of (10), since [2, p. 203]

$$\int_0^\infty \sin\left(t\sqrt{p^2+\sigma}\,\right)\frac{pJ_0(pr)}{\sqrt{p^2+\sigma}}dp = \begin{cases} 0 & \text{for} \quad r>t \\ \left(t^2-r^2\right)^{-1/2}\cos\left(\sqrt{\left(t^2-r^2\right)\sigma}\,\right) & \text{for} \quad r<t. \end{cases}$$

Substituting (11) into (9) and following the lines of [1], for the transform $\bar{\varphi}(t,s)$, we obtain the integral equation

$$\int_0^r \frac{t\bar{\varphi}(t,s)}{\sqrt{r^2-t^2}}\exp\left(-\sqrt{\left(r^2-t^2\right)\sigma}\,\right)dt + \int_0^R \bar{\varphi}(t,s)\sin\left(t\sqrt{\sigma}\,\right)dt - \int_r^R \frac{t\bar{\varphi}(t,s)}{\sqrt{t^2-r^2}}\sin\left(\sqrt{\left(t^2-r^2\right)\sigma}\,\right)dt$$
$$= \frac{\bar{q}_1(s)}{\lambda}\int_0^r q_2(\varrho)\varrho\,d\varrho, \qquad 0<r<R.$$

Applying the integrating factor $2\mu\cos\left(\sqrt{\left(r^2-\mu^2\right)\sigma}\,\right)/\sqrt{r^2-\mu^2}$ and performing manipulations similar to the solution of the Abel equation in [3, p. 46], we arrive at the integral equation

$$\bar{\varphi}(t,s) - \frac{1}{\pi}\int_0^R \bar{\varphi}(\varrho,s)\left[\frac{\sin\left((\varrho-t)\sqrt{\sigma}\,\right)}{\varrho-t} - \frac{\sin\left((\varrho+t)\sqrt{\sigma}\,\right)}{\varrho+t}\right]d\varrho$$
$$= \frac{2\bar{q}_1(s)}{\pi\lambda}\int_0^t \frac{q_2(\mu)\cos\left(\sqrt{\left(t^2-\mu^2\right)\sigma}\,\right)\mu}{\sqrt{t^2-\mu^2}}d\mu, \qquad 0<t<R,$$

or

$$\bar{\psi}(t,s) - \frac{1}{\pi}\int_0^R \bar{\psi}(\varrho,s)\left[\frac{\sin\left((\varrho-t)\sqrt{\sigma}\,\right)}{\varrho-t} - \frac{\sin\left((\varrho+t)\sqrt{\sigma}\,\right)}{\varrho+t}\right]d\varrho$$
$$= \frac{2}{\pi\lambda s}\int_0^t \frac{q_2(\mu)\cos\left(\sqrt{\left(t^2-\mu^2\right)\sigma}\,\right)\mu}{\sqrt{t^2-\mu^2}}d\mu, \qquad 0<t<R,$$

where $\bar{\psi}(\varrho, s) := \bar{\varphi}(\varrho, s)/\left(s\bar{q}_1(s)\right)$. There is no ready-to-use method other than the approach of [4] for the solution of the last integral equation in the domain of $L$-transforms.

We represent the unknown function $\bar{\psi}(t, s)$ in the form of the series

$$\begin{aligned}\bar{\psi}(t, s) &= \exp\left(-R\sqrt{\sigma}\,\right)\sum_{n=0}^{\infty}\psi_n(t)\left(\sqrt{s}\,\right)^{n-2} \\ &= \frac{1}{s}\exp\left(-R\sqrt{\sigma}\,\right)\psi_0(t) + \exp\left(-R\sqrt{\sigma}\,\right)\sum_{n=0}^{\infty}\psi_{n+1}(t)\left(\sqrt{s}\,\right)^{n-1},\end{aligned} \tag{12}$$

which justifies the existence of the inverse Laplace transform $L^{-1}\left[\bar{\psi}(t, s)\right]$, since

$$L^{-1}\left[s^{-1}\exp\left(-k\sqrt{s}\,\right)\right] = \operatorname{erfc}\left(k/\left(2\sqrt{\tau}\,\right)\right),$$

$$L^{-1}\left[\left(\sqrt{s}\,\right)^{m-1}\exp\left(-k\sqrt{s}\,\right)\right] = \left(\exp\left(-k^2/(4\tau)\right)/\left(2^m\sqrt{\pi\tau^{m+1}}\,\right)\right)H_m\left(k/\left(2\sqrt{\tau}\,\right)\right),$$

where

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty}\exp\left(-t^2\right)dt = 1 - \operatorname{erf}(x) = 1 - \frac{2}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{k!(2k+1)},$$

$H_m(x) = m!\sum_{k=0}^{[m/2]}(-1)^k(2x)^{m-2k}/(k!(m-2k)!)$ is the Hermite polynomial [5].

Following [4], we construct a recursion formula for the functions $\psi_n(t)$ occurring in (12):

$$\psi_n(t) = \frac{2}{\pi\lambda}\sum_{j=0}^{n}A_{nj}(R)\int_0^t\left(\sqrt{t^2-\mu^2}\,\right)^{j-1}q_2(\mu)\mu\,d\mu + \frac{1}{\pi}\sum_{m=0}^{n}\int_0^R C_m(\varrho, t)\psi_{n-m}(\varrho)d\varrho, \tag{13}$$

$$\psi_0(t) = \frac{2}{\pi\lambda}\int_0^t\frac{q_2(\mu)\mu}{\sqrt{t^2-\mu^2}}d\mu, \qquad A_{nj}(R) = \frac{1}{n!}\left(\frac{1}{\sqrt{a}}\right)^n\binom{n}{j}R^{n-j}\cos\left(\frac{j\pi}{2}\right),$$

$$C_m(\varrho, t) = \frac{1}{m!}\left(\frac{1}{\sqrt{a}}\right)^m\sin\left(\frac{m\pi}{2}\right)\left[(\varrho-t)^{m-1} - (\varrho+t)^{m-1}\right],$$

where $\binom{n}{j}$ are the binomial coefficients.

Successively substituting the expressions (13) into (12) and the resulting relations into (11) and finally into (7), we can find the solution of the problem in the domain of $L$-transforms:

$$\begin{aligned}\bar{T}(r, z, s) = {}& \bar{q}_1(s)\exp\left(-R\sqrt{\sigma}\,\right)\int_0^R\psi_0(t)\int_0^{\infty}\exp\left(-z\sqrt{p^2+\sigma}\,\right)\sin\left(t\sqrt{p^2+\sigma}\,\right)\frac{pJ_0(pr)}{\sqrt{p^2+\sigma}}dp\,dt \\ & + \bar{q}_1(s)\exp\left(-R\sqrt{\sigma}\,\right)\sum_{n=0}^{\infty}\left(\sqrt{s}\,\right)^{n+1}\sum_{j=0}^{n+1}\left[\frac{2}{\pi\lambda}A_{n+1\,j}(R)\int_0^R\int_0^t\left(\sqrt{t^2-\mu^2}\,\right)^{j-1}q_2(\mu)\mu\right. \\ & \times\int_0^{\infty}\exp\left(-z\sqrt{p^2+\sigma}\,\right)\sin\left(t\sqrt{p^2+\sigma}\,\right)\frac{pJ_0(pr)}{\sqrt{p^2+\sigma}}dp\,d\mu\,dt \\ & + \sum_{m=0}^{n+1}\frac{1}{\pi}\int_0^R\int_0^R C_m(\varrho, t)\psi_{n+1-m}(\varrho)\int_0^{\infty}\exp\left(-z\sqrt{p^2+\sigma}\,\right)\sin\left(t\sqrt{p^2+\sigma}\,\right) \\ & \left.\times\frac{pJ_0(pr)}{\sqrt{p^2+\sigma}}dp\,d\varrho\,dt\right],\end{aligned} \tag{14}$$

where $r > 0$, $z > 0$, and $\operatorname{Re} s > 0$. We can readily see that this solution satisfies the mixed boundary conditions (5) and (6) for $z = 0$.

In the special case $q_1(\tau)q_2(r) = \text{const}$, the inverse transform $T(r, z, \tau)$ of the solution was found in [1, 4]. For general $q_1(\tau)q_2(r)$, the inverse transform of the solution can be found with the use of the inverse formula for the Laplace integral [5, p. 807 of the Russian translation; 6, p. 499]:

$$T(r, z, \tau) = \frac{1}{2\pi i} \int\limits_{\alpha - i\infty}^{\alpha + i\infty} \exp(\tau s) \bar{T}(r, z, s) ds.$$

Note that the solution (14) satisfies the well-known limit relations as $s \to 0$ $(\tau \to \infty)$ and as $s \to \infty$ $(\tau \to 0)$ provided that the limits $\lim_{s \to 0} [s\bar{q}_1(s)]$ and $\lim_{s \to \infty} [s\bar{q}_1(s)]$ exist.

## REFERENCES

1. Kozlov, V.P., Yurchuk, N.I., and Mandrik, P.A., *I.F.Zh.*, 1998, vol. 71, no. 4, pp. 734–743.
2. Prudnikov, A.P., Brychkov, Yu.A., and Marichev, O.I., *Integraly i ryady. Spetsial'nye funktsii* (Integrals and Series. Special Functions), Moscow, 1983.
3. Mikhlin, S.G., *Lektsii po lineinym integral'nym uravneniyam* (Lectures on Linear Integral Equations), Moscow, 1959.
4. Kozlov, V.P., Yurchuk, N.I., and Mandrik, P.A., *Vestn. Bel. univ. Ser. 1*, 1999, no. 2, pp. 37–42.
5. Abramowitz, M. and Stegun, I., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Washington, 1964. Translated under the title *Spravochnik po spetsial'nym funktsiyam*, Moscow, 1979.
6. Lykov, A.V., *Teoriya teploprovodnosti* (Theory of Thermal Conduction), Moscow, 1967.