Differential Equations, Vol. 37, No. 8, 2001, pp. 1171–1175. Translated from Differentsial'nye Uravneniya, Vol. 37, No. 8, 2001, pp. 1118–1121.

SHORT COMMUNICATIONS

Method for Solving Nonstationary Heat Problems with Mixed Discontinuous Boundary Conditions on the Boundary of a Half-Space

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Received January 25, 2001

Suppose that we must find a solution of the differential nonstationary ($\tau > 0$) heat equation

$$\theta_{rr}(r,z,\tau) + r^{-1}\theta_r(r,z,\tau) + \theta_{zz}(r,z,\tau) = a^{-1}\theta_\tau(r,z,\tau) \tag{1}$$

in the cylindrical coordinates ($r, z > 0$) for a half-space with the homogeneous initial condition

$$\theta(r,z,0) = 0 \tag{2}$$

and the mixed boundary conditions

$$\theta(r,0,\tau) = f_1(r)f_2(\tau), \qquad 0 < r < R, \qquad \theta_z(r,0,\tau) = 0, \qquad R < r < \infty, \tag{3}$$

on the surface $z = 0$ in the case of the axial symmetry $\theta_r(0,z,\tau) = 0$.

If to problem (1)–(3), we apply the Laplace integral transform (the L-transform)

$$\bar\theta(r,z,s) = L[\theta(r,z,\tau)] = \int_0^\infty \theta(r,z,\tau)\exp(-s\tau)d\tau, \qquad \operatorname{Re} s > 0,$$

then the problem acquires the form

$$\bar\theta_{rr}(r,z,s) + r^{-1}\bar\theta_r(r,z,s) + \bar\theta_{zz}(r,z,s) = \sigma\bar\theta(r,z,s), \qquad \sigma = s/a, \tag{4}$$

$$\bar\theta(r,0,s) = f_1(r)\bar f_2(s), \qquad 0 < r < R, \qquad \bar\theta_z(r,0,s) = 0, \qquad R < r < \infty, \tag{5}$$

where

$$\bar f_2(s) = \int_0^\infty f_2(\tau)\exp(-s\tau)d\tau. \tag{6}$$

Theorem. *Suppose that the integral* (6) *exists, the real part of the parameter* s *is positive* (*i.e.*, $\operatorname{Re} s > 0$), *and the temperature remains bounded as* $\sqrt{r^2+z^2} \to \infty$. *Then the solution of the problem can be represented via* L*-transforms in the form*

$$\begin{aligned}\bar\theta(r,0,s) = {} & \frac{2\bar f_2(s)}{\pi}\exp\left(-R\sqrt{\sigma}\right)\int_0^\infty \exp\left(-z\sqrt{p^2+\sigma}\right)\frac{pJ_0(pr)dp}{\sqrt{p^2+\sigma}} \\ & \times \int_0^R \cos\left(t\sqrt{p^2+\sigma}\right)\left[\frac{d}{dt}\int_0^t F_1(t,\mu)\right]dt \\ & + \frac{\bar f_2(s)}{\pi}\exp\left(-R\sqrt{\sigma}\right)\int_0^\infty \exp\left(-z\sqrt{p^2+\sigma}\right)\frac{pJ_0(pr)dp}{\sqrt{p^2+\sigma}}\sum_{k=0}^\infty s^{(k+1)/2}\int_0^R \cos\left(t\sqrt{p^2+\sigma}\right) \\ & \times \left[\frac{d}{dt}\int_0^t D_{k+1}(R,t,\mu)F_1(t,\mu)d\mu + \sum_{m=0}^{k+1}\int_0^R C_m(x,t)\varphi_{k-m+1}(x)dx\right]dt,\end{aligned} \tag{7}$$

where

$$C_m(x,t) = (1/m!)\left(\sqrt{a}\,\right)^{-m}\sin(m\pi/2)\left[(x-t)^{m-1} + (x+t)^{m-1}\right], \tag{8}$$

$$D_n(R,t,\mu) = \frac{2}{n!}\left(\sqrt{a}\,\right)^{-n}\sum_{j=0}^{n}\binom{n}{j}R^{n-j}\cos\left(\frac{j\pi}{2}\right)\left(\sqrt{t^2-\mu^2}\,\right)^{j}, \tag{9}$$

$F_1(t,\mu) = f_1(\mu)\mu/\sqrt{t^2-\mu^2}$, *the* $\binom{n}{j}$ *are the binomial coefficients, and* $J_0(pr)$ *is the Bessel function of the first kind.*

Proof. The solution of Eq. (4) has the form [1]

$$\bar\theta(r,z,s) = \int_0^\infty \bar C(p,s)\exp\left(-z\sqrt{p^2+\sigma}\,\right)J_0(pr)\,dp, \tag{10}$$

where $\bar C(p,s)$ is the unknown function for which the following dual integral equations in terms of L-transforms can be obtained from (5):

$$\begin{aligned}\int_0^\infty \bar C(p,s)J_0(pr)\,dp &= f_1(r)\bar f_2(s), \qquad 0<r<R,\\ \int_0^\infty \bar C(p,s)\sqrt{p^2+\sigma}\,J_0(pr)\,dp &= 0, \qquad R<r<\infty.\end{aligned} \tag{11}$$

If we introduce a new unknown function $\bar\varphi(t,s)$ by the formula

$$\bar C(p,s) = \frac{p}{\sqrt{p^2+\sigma}}\int_0^R \bar\varphi(t,s)\cos\left(t\sqrt{p^2+\sigma}\,\right)dt, \tag{12}$$

then, using the substitution (12), we can readily show that the second dual integral equation in (11) is automatically valid with regard for the following well-known value of the discontinuous integral [2, p. 203]:

$$\int_0^\infty \frac{pJ_0(pr)}{\sqrt{p^2+\sigma}}\sin\left(x\sqrt{p^2+\sigma}\,\right)dp = \begin{cases} 0 & \text{if } x<r\\ \cos\left(\sqrt{(x^2-r^2)\,\sigma}\,\right)\left(x^2-r^2\right)^{-1/2} & \text{if } x>r.\end{cases}$$

The substitution of (12) into the first equation in (11) gives the integral equation

$$\int_0^r \frac{\bar\varphi(t,s)}{\sqrt{r^2-t^2}}\exp\left(-\sqrt{(r^2-t^2)\,\sigma}\,\right)dt - \int_r^R \frac{\bar\varphi(t,s)}{\sqrt{t^2-r^2}}\sin\left(\sqrt{(t^2-r^2)\,\sigma}\,\right)dt = f_1(r)\bar f_2(s), \quad 0<r<R, \tag{13}$$

with regard for the following well-known value of the discontinuous integral [2, p. 203]:

$$\int_0^\infty \frac{pJ_0(pr)}{\sqrt{p^2+\sigma}}\cos\left(t\sqrt{p^2+\sigma}\,\right)dp = \begin{cases} \exp\left(-\sqrt{(r^2-t^2)\,\sigma}\,\right)\left(r^2-t^2\right)^{-1/2} & \text{if } t<r\\ \sin\left(-\sqrt{(t^2-r^2)\,\sigma}\,\right)\left(t^2-r^2\right)^{-1/2} & \text{if } t>r.\end{cases}$$

The integral equation (13) in terms of L-transforms is a basis for finding the unknown function $\bar{\varphi}(r,s)$. We reduce it to a Fredholm integral equation of the second kind (but with the parameter s). To this end, we divide both sides of Eq. (13) by $\bar{f}_2(s) \neq 0$, replace r by μ, and multiply the resulting relation by the integrating factor $2\mu\cos\left(\sqrt{(r^2-\mu^2)\,\sigma}\right)(r^2-\mu^2)^{-1/2}$. Further, integrating both sides of the equation with respect to μ from 0 to r, we obtain

$$\int_0^r \frac{\cos\left(\sqrt{(r^2-\mu^2)\,\sigma}\right)}{\sqrt{(r^2-\mu^2)}}\mu \int_0^\mu \frac{\bar{\varphi}^*(t,s)}{\sqrt{\mu^2-t^2}} \exp\left(-\sqrt{(\mu^2-t^2)\,\sigma}\right) dt\, d\mu$$
$$- \int_0^r \frac{\cos\left(\sqrt{(r^2-\mu^2)\,\sigma}\right)}{\sqrt{r^2-\mu^2}}\mu \int_\mu^R \frac{\bar{\varphi}^*(t,s)}{\sqrt{t^2-\mu^2}} \sin\left(\sqrt{(t^2-\mu^2)\,\sigma}\right) dt\, d\mu$$
$$= \int_0^r \frac{f_1(\mu)\cos\left(\sqrt{(r^2-\mu^2)\,\sigma}\right)}{\sqrt{r^2-\mu^2}}\mu\, d\mu, \qquad 0 < r < R,$$

where

$$\bar{\varphi}^*(t,s) = \bar{\varphi}(t,s)/\left(s\bar{f}_2(s)\right). \tag{14}$$

Changing the order of integration on the left-hand side of the last relation and computing the resulting integrals, we arrive at the integral equation

$$\int_0^r \bar{\varphi}^*(t,s)dt - \frac{1}{\pi}\int_0^R \bar{\varphi}^*(t,s)\left[\operatorname{si}\left((t+r)\sqrt{\sigma}\,\right) - \operatorname{si}\left((t-r)\sqrt{\sigma}\,\right)\right] dt$$
$$= \frac{2}{\pi s}\int_0^r \frac{f_1(\mu)\cos\left(\sqrt{(r^2-\mu^2)\,\sigma}\right)}{\sqrt{r^2-\mu^2}}\mu\, d\mu, \qquad 0 < r < R,$$

where $\operatorname{si}(x)$ is the sine integral [3, p. 59 of the Russian translation]. Hence, differentiating both sides with respect to r, we obtain the following integral equation in terms of L-transforms for the function $\bar{\varphi}^*(r,s)$:

$$\bar{\varphi}^*(r,s) - \frac{1}{\pi}\int_0^R \bar{\varphi}^*(t,s)\bar{K}(r,t,s)dt = \bar{F}(r,s), \qquad 0 < r < R, \tag{15}$$

where

$$\bar{K}(r,t,s) = \frac{\sin\left((t-r)\sqrt{\sigma}\,\right)}{t-r} + \frac{\sin\left((t+r)\sqrt{\sigma}\,\right)}{t+r}, \tag{16}$$

$$\bar{F}(r,s) = \frac{2}{s\pi}\frac{d}{dr}\int_0^r \frac{f_1(\mu)\cos\left(\sqrt{(r^2-\mu^2)\,\sigma}\right)}{\sqrt{r^2-\mu^2}}\mu\, d\mu. \tag{17}$$

Note that the solution of problem (15)–(17) with $f_1(\mu) \equiv 1$ was found in [4]. Here we find the solution in the case of a general function $f_1(\mu)$.

We represent the unknown analytic function $\bar{\varphi}^*(r,s)$ as the function series

$$\bar{\varphi}^*(r,s) = \frac{1}{s}\exp\left(-R\sqrt{\sigma}\,\right)\sum_{n=0}^{\infty}\varphi_n(r)\left(\sqrt{s}\,\right)^n, \qquad 0 < r < R, \tag{18}$$

where the $\varphi_n(r)$ are some auxiliary functions. The kernel $\bar{K}(r,t,s)$ given by (16) can be represented as

$$\bar{K}(r,t,s) = \sum_{k=0}^{\infty} C_k(t,r)\left(\sqrt{s}\,\right)^k,$$

and

$$\exp\left(R\sqrt{\sigma}\,\right)\cos\left(\sqrt{(r^2-\mu^2)\,\sigma}\right) = \frac{1}{2}\sum_{n=0}^{\infty} D_n(R,r,\mu)\left(\sqrt{s}\,\right)^n,$$

where $C_k(t,r)$ and $D_n(R,r,\mu)$ are given by (8) and (9), respectively.

Substituting these expressions into (15) and performing related multiplications of series, we arrive at the relation

$$\pi\sum_{n=0}^{\infty}\varphi_n(r)\left(\sqrt{s}\,\right)^{n-2} = \sum_{n=0}^{\infty}\left[\frac{d}{dr}\int_0^r D_n(R,r,\mu)f_1(\mu)\frac{\mu\,d\mu}{\sqrt{r^2-\mu^2}} + \sum_{m=0}^{n}\int_0^R C_m(t,r)\varphi_{n-m}(t)\,dt\right]\left(\sqrt{s}\,\right)^{n-2},$$

$0<r<R$, which is not an integral equation for $\varphi_{n-m}(t)$, since, obviously, $C_0(t,r)\equiv 0$.

Therefore, for the unknown auxiliary functions $\varphi_n(r)$, we can readily write out the recursion formula

$$\varphi_n(r) = \frac{1}{\pi}\frac{d}{dr}\int_0^r D_n(R,r,\mu)\frac{f_1(\mu)\mu\,d\mu}{\sqrt{r^2-\mu^2}} + \frac{1}{\pi}\sum_{m=0}^{n}\int_0^R C_m(t,r)\varphi_{n-m}(t)\,dt, \qquad 0<r<R,$$

which, in particular, implies that

$$\varphi_0(r) = \frac{2}{\pi}\frac{d}{dr}\int_0^r \frac{f_1(\mu)\mu\,d\mu}{\sqrt{r^2-\mu^2}}, \qquad \varphi_1(r) = \frac{2R}{\pi\sqrt{a}}\frac{d}{dr}\int_0^r \frac{f_1(\mu)\mu\,d\mu}{\sqrt{r^2-\mu^2}} + \frac{4}{\pi^2\sqrt{a}}\int_0^R \frac{f_1(\mu)\mu\,d\mu}{\sqrt{R^2-\mu^2}},$$

and so on.

Substituting the expression for $\varphi_n(r)$ into (18), we can find the transform $\bar{\varphi}^*(r,s)$, for which the inverse Laplace transform exists, since

$$L^{-1}\left[\frac{1}{s}\exp\left(-R\sqrt{\sigma}\,\right)\right] = \operatorname{erfc}\left(\frac{R}{2\sqrt{a\tau}}\right),$$

$$L^{-1}\left[s^{(k-1)/2}\exp\left(-R\sqrt{\sigma}\,\right)\right] = \frac{1}{2^k\sqrt{\pi\tau^{k+1}}}\exp\left(-\frac{R^2}{4a\tau}\right)H_k\left(\frac{R}{2\sqrt{a\tau}}\right),$$

where $\operatorname{erfc}(x) = (2/\sqrt{\pi}\,)\int_x^{\infty}\exp(-t^2)\,dt$ and $H_k(x)$ is the Hermite polynomial [3, p. 579 of the Russian translation]. Performing related substitutions, we obtain

$$\bar{\varphi}(r,s) = \frac{2\bar{f}_2(s)}{\pi}\exp\left(-R\sqrt{\sigma}\,\right)\frac{d}{dr}\int_0^r\frac{f_1(\mu)\mu\,d\mu}{\sqrt{r^2-\mu^2}} + \frac{\bar{f}_2(s)}{\pi}\exp\left(-R\sqrt{\sigma}\,\right)\sum_{k=0}^{\infty}s^{(k+1)/2}\left[\frac{d}{dr}\int_0^r D_{k+1}(R,r,\mu)\frac{f_1(\mu)\mu\,d\mu}{\sqrt{r^2-\mu^2}} + \sum_{m=0}^{k+1}\int_0^R C_m(t,r)\varphi_{k-m+1}(t)dt\right].$$

Then, using (12) and (10), we can write out the solution of the original problem in terms of L-transforms in the form (7), which, together with the inversion formula for the Laplace integral [5, p. 205 of the Russian translation], gives $\theta(r, s, \tau)$. This completes the proof of the theorem.

Note that if $f_1(\mu) = T_c - T_0 = \text{const} \neq 0$, where T_0 is the initial temperature of the half-space and T_c is the temperature on the surface $z = 0$ in the disk $0 < r < R$, then the inner integrals in the solution (7) can readily be evaluated:

$$\int_0^R \cos\left(t\sqrt{p^2+\sigma}\right)\left[\frac{d}{dt}\int_0^t \frac{(T_c - T_0)\,\mu\,d\mu}{\sqrt{t^2-\mu^2}}\right] dt = (T_c - T_0)\,\frac{\sin\left(R\sqrt{p^2+\sigma}\right)}{\sqrt{p^2+\sigma}},$$

$$\int_0^R \cos\left(t\sqrt{p^2+\sigma}\right)\left[\frac{d}{dt}\int_0^t D_{k+1}(R,t,\mu)\frac{(T_c - T_0)\,\mu\,d\mu}{\sqrt{t^2-\mu^2}}\right] dt$$

$$= (T_c - T_0)\sum_{j=0}^{k+1}\sum_{n=0}^{j}\frac{(j+1)n!}{(k+1)!}\binom{k+1}{j}\binom{j}{n}B\left(1,\frac{j+1}{2}\right)\left(\frac{R}{\sqrt{a}}\right)^{k+1}$$

$$\times \cos\left(\frac{j\pi}{2}\right)\frac{\sin\left(R\sqrt{p^2+\sigma}+n\pi/2\right)}{R^n\left(\sqrt{p^2+\sigma}\right)^{n+1}},$$

where $B(\alpha, \beta)$ is the beta function [6, p. 24].

If $s \to 0$ ($\tau \to \infty$), then

$$\lim_{s\to 0}\left[s\bar{\varphi}(r,s)\right] = \frac{2}{\pi}\frac{d}{dr}\int_0^r \frac{f_1(\mu)\mu\,d\mu}{\sqrt{r^2-\mu^2}}\lim_{s\to 0}\left[s\bar{f}_2(s)\right];$$

in particular, if $f_1(\mu) = T_c - T_0$, then we have $\lim_{s\to 0}\left[s\bar{\varphi}(r,s)\right] = (2/\pi)\,(T_c - T_0)$, which coincides with the similar solutions obtained in [7, 8] for the corresponding stationary Laplace equation with mixed boundary conditions.

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