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SHORT COMMUNICATIONS

Solution of a Heat Equation of Hyperbolic Type with Mixed Boundary Conditions on the Surface of an Isotropic Half-Space

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INTRODUCTION

The phenomenological theory of heat conduction assumes that the heat propagation velocity is infinitely large, which is justified by computations of temperature fields in bodies under ordinary conditions observed in practice. However, for rarefied media and nonstationary processes of high intensity, one should take account of the fact that heat propagates at a finite velocity. Lykov [1, p. 21] suggested a theory of finite heat and particle propagation velocities in capillary-porous bodies, which implies that $w_r = \sqrt{\lambda/(c\gamma\tau_r)}$, where w_r is the heat propagation velocity, τ_r is the time constant of the thermal flow, λ is the thermal conductivity, c is the specific heat, and γ is the material density. As a rule, $w_r \cong 150 - 300$ m/sec and $\tau_r \cong 10^{-9} - 10^{-15}$ sec for gases, and the effect of the finite heat propagation velocity on the heat transfer becomes observable in the case of a supersonic flow. Then the heat equation has the form [1, p. 21]

$$\mathbf{q} = -\lambda \nabla T - \tau_r \partial \mathbf{q}/\partial \tau, \tag{1}$$

where $\mathbf{q}$ is the heat flux density vector, T is temperature, and ∇ is the gradient operator.

STATEMENT OF THE PROBLEM

Suppose that we wish to find the evolution laws for the spatial temperature fields in an isotropic half-space under mixed boundary conditions on the surface, where the discontinuity line is a circle of radius R. According to (1), in the cylindrical coordinates $r \geq 0$, $z \geq 0$, the axisymmetric heat equation of hyperbolic type has the form [1, p. 21]

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T(r,z,\tau)}{\partial r}\right) + \frac{\partial^2 T(r,z,\tau)}{\partial z^2} = \frac{1}{a}\frac{\partial T(r,z,\tau)}{\partial \tau} + \frac{1}{w_r^2}\frac{\partial^2 T(r,z,\tau)}{\partial \tau^2}, \qquad r > 0, \qquad z > 0, \tag{2}$$

where $a > 0$ is the temperature conductivity and $\tau > 0$ is the time variable.

We assume that the thermal characteristics (a, λ, c) are independent of temperature in a given temperature range, and we deal with their average values; this approximation can always be justified if appropriate new variables are introduced (e.g., see [2, p. 15 of the Russian translation]).

The boundary conditions at infinity have the form

$$\partial T(r,\infty,\tau)/\partial z = \partial T(\infty,z,\tau)/\partial r = 0, \qquad r > 0, \qquad z > 0, \tag{3}$$

the symmetry condition is

$$\partial T(0,z,\tau)/\partial r = 0, \qquad z > 0, \tag{4}$$

and the mixed boundary conditions on the surface $z = 0$ have the form

$$-\partial T(r,0,\tau)/\partial z = \lambda^{-1} q(\tau) + C^{-1} w_r^{-2} \partial q(\tau)/\partial \tau, \qquad 0 < r < R, \tag{5}$$

$$T(r,0,\tau) = 0, \qquad R < r < \infty, \tag{6}$$

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where $C = c\gamma$ is the thermal capacity per unit volume and $q(\tau)$ is the heat flux density in the domain $0 < r < R$ on the surface $z = 0$ of the half-space.

Let us consider the homogeneous initial conditions for Eqs. (2) and (5):

$$T(r,z,0) = \partial T(r,z,0)/\partial\tau = 0, \qquad r > 0, \qquad z > 0, \tag{7}$$

$$q(0) = 0. \tag{8}$$

By applying the integral Laplace transform (e.g., see [3, p. 63])

$$\bar{T}(r,z,s) = L[T(r,z,\tau)] = \int_0^\infty \exp(-s\tau)T(r,z,\tau)d\tau, \qquad \operatorname{Re} s > 0,$$

$$\bar{q}(s) = L[q(\tau)] = \int_0^\infty \exp(-s\tau)q(\tau)d\tau, \qquad \operatorname{Re} s > 0,$$

we reduce problem (2)–(8) to the problem

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\bar{T}(r,z,s)}{\partial r}\right) + \frac{\partial^2\bar{T}(r,z,s)}{\partial z^2} = s^*\bar{T}(r,z,s), \qquad r > 0, \qquad z > 0; \tag{9}$$

$$\frac{\partial\bar{T}(r,\infty,s)}{\partial z} = \frac{\partial\bar{T}(\infty,z,s)}{\partial r} = \frac{\partial\bar{T}(0,z,s)}{\partial r} = 0, \qquad r > 0, \qquad z > 0; \tag{10}$$

$$-\frac{\partial\bar{T}(r,0,s)}{\partial z} = \bar{q}(s)\left(\lambda^{-1} + C^{-1}w_r^{-2}s\right), \qquad 0 < r < R; \tag{11}$$

$$\bar{T}(r,0,s) = 0, \qquad R < r < \infty, \tag{12}$$

for Laplace transforms, where $s^* = w_r^{-2}s^2 + a^{-1}s$; from now on, the condition $\operatorname{Re} s > 0$ on the L-parameter is omitted for brevity.

Note that in the general case of nonhomogeneous initial conditions [unlike the homogeneous conditions (7) and (8)], the corresponding Laplace transforms of the right-hand sides of (2) and (5) have the form

$$L\left[a^{-1}\frac{\partial T(r,z,\tau)}{\partial\tau} + w_r^{-2}\frac{\partial^2 T(r,z,\tau)}{\partial\tau^2}\right] = s^*\bar{T}(r,z,s) - \left(w_r^{-2}s + a^{-1}\right)T(r,z,0) - w_r^{-2}\frac{\partial T(r,z,0)}{\partial\tau},$$

$$L\left[\lambda^{-1}q(\tau) + C^{-1}w_r^{-2}\frac{\partial q(\tau)}{\partial\tau}\right] = \left(w_r^{-2}s + a^{-1}\right)\bar{q}(s) - C^{-1}w_r^{-2}q(0).$$

Without loss of generality, in the following, we consider the case of the homogeneous initial conditions (7) and (8).

Assertion. *The solution of the original heat transfer problem* (2)–(8) *can be expressed as*

$$T(r,z,\tau) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\exp(s\tau)\int_0^\infty pJ_0(pr)\frac{\exp(-z\psi(p,s))}{\psi(p,s)}\int_0^R \bar{\varphi}(t,s)\sin(t\psi(p,s))dt\,dp\,ds, \tag{13}$$
$$r \ge 0, \qquad z \ge 0, \qquad \tau > 0, \qquad \operatorname{Re} s > \sigma > 0,$$

where $\psi(p,s) = \sqrt{p^2 + w_r^{-2}s^2 + a^{-1}s}$.

Proof. If we apply the infinite integral Hankel transform (e.g., see [3, p. 64])

$$\bar{T}_H(p,z,s) = H\left[\bar{T}(r,z,s)\right] = \int_0^\infty \bar{T}(r,z,s)J_0(pr)r\,dr$$

to Eq. (9), where $J_0(pr)$ is the Bessel function of a real argument, then we obtain the ordinary differential equation

$$d^2\bar{T}_H(p,z,s)/dz^2 - \left(p^2+s^*\right)\bar{T}_H(p,z,s) = 0, \qquad z>0,$$

whose particular solution, in view of the boundary conditions (10), can be rewritten in the form [3, p. 166] $\bar{T}_H(p,z,s) = \bar{D}(p,s)\exp\left(-z\sqrt{p^2+s^*}\right)$, $z>0$, where $\bar{D}(p,s)$ is an unknown analytic function to be determined.

By applying the inversion formula for the Hankel transform, we find a solution of the problem in terms of L-transforms in the form

$$\bar{T}(r,z,s) = \int_0^\infty \bar{D}(p,s)\exp\left(-z\left(p^2+s^*\right)^{1/2}\right)J_0(pr)p\,dp, \qquad r>0, \qquad z>0. \tag{14}$$

Using the mixed boundary conditions (11) and (12), from the expression (14) with $z=0$ for the unknown function $\bar{D}(p,s)$, we obtain the paired integral equations with the L-parameter:

$$\begin{gathered}\int_0^\infty \bar{D}(p,s)\left(p^2+s^*\right)^{1/2}J_0(pr)p\,dp = \bar{q}(s)\left(\lambda^{-1}+C^{-1}w_r^{-2}s\right), \qquad 0<r<R,\\ \int_0^\infty \bar{D}(p,s)J_0(pr)p\,dp = 0, \qquad R<r<\infty,\end{gathered} \tag{15}$$

where the existence and convergence of the corresponding integral of $\bar{q}(s)$ are assumed.

The solution of the paired integral equations (15) can be found with the use of the ansatz

$$\bar{D}(p,s) = \left(p^2+s^*\right)^{-1/2}\int_0^R \bar{\varphi}(t,s)\sin\left(t\left(p^2+s^*\right)^{1/2}\right)dt, \tag{16}$$

where $\bar{\varphi}(t,s)$ is an auxiliary unknown mapping function. With the ansatz (16), the second equation in (15) is satisfied automatically, regardless of the choice of the function $\bar{\varphi}(t,s)$, since the corresponding discontinuous integral vanishes for $R<r<\infty$:

$$\begin{aligned}&\int_0^R \left(p^2+s^*\right)^{-1/2}\sin\left(t\left(p^2+s^*\right)^{1/2}\right)J_0(pr)p\,dp\\ &\qquad = \begin{cases} 0 & \text{if } \ 0<t<r\\ \left(t^2-r^2\right)^{-1/2}\cos\left(t\left(s^*\left(t^2-r^2\right)\right)^{1/2}\right) & \text{if } \ 0<r<t, \ \operatorname{Re}\sqrt{s^*}>0.\end{cases}\end{aligned}$$

If we substitute (16) into the first equation in (15), use the relation

$$pJ_0(pr) = r^{-1}d\left(rJ_1(pr)\right)/dr,$$

and integrate the resulting relation with respect to r from 0 to r, then we obtain an equation for the auxiliary analytic function $\bar{\varphi}(t,s)$:

$$\begin{aligned}&\int_0^r \left(r^2-t^2\right)^{-1/2}t\bar{\varphi}(t,s)\exp\left(-\left(s^*\left(r^2-t^2\right)\right)^{1/2}\right)dt\\ &\qquad -\int_r^R \left(t^2-r^2\right)^{-1/2}t\bar{\varphi}(t,s)\sin\left(\left(s^*\left(t^2-r^2\right)\right)^{1/2}\right)dt + \int_0^R \bar{\varphi}(t,s)\sin\left(t\sqrt{s^*}\right)dt\\ &\quad = 2^{-1}\bar{q}(s)r^2\left(\lambda^{-1}+C^{-1}w_r^{-2}s\right), \qquad 0<r<R.\end{aligned} \tag{17}$$

It is noteworthy that methods for finding the analytic function $\bar{\varphi}(t,s)$ in equations like (17) were described, e.g., in [3, p. 196], and in the derivation of (17), we have used the value of the discontinuous integral

$$\int\limits_0^\infty \sin\left(t\left(p^2+s^*\right)^{1/2}\right)J_1(pr)dp$$
$$=\begin{cases} r^{-1}\left[\sin\left(t\sqrt{s^*}\,\right)+t\left(r^2-t^2\right)^{-1/2}\exp\left(-\left(s^*\left(r^2-t^2\right)\right)^{1/2}\right)\right] & \text{if} \quad 0<t<r \\ r^{-1}\left[\sin\left(t\sqrt{s^*}\,\right)-t\left(t^2-r^2\right)^{-1/2}\sin\left(\left(s^*\left(t^2-r^2\right)\right)^{1/2}\right)\right] & \text{if} \quad 0<r<t,\ \operatorname{Re}\sqrt{s^*}>0. \end{cases}$$

Finally, after returning to Eqs. (14) and (16) and applying the inverse Laplace transform, we have the solution of the original problem in the form (13). The proof of the assertion is complete.

If the heat propagation velocity tends to infinity, $w_r \to \infty$, then formula (13) degenerates into the solution of the parabolic heat equation for the corresponding mixed boundary conditions.

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