NONLINEAR FILTERING OF TIME SERIES USING THE WAVELET TRANSFORM

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Abstract

A fast nonlinear filtering algorithm is presented. This algorithm propagates the entire conditional probability functions recursively in a computationally efficient manner using the discrete wavelet transform. With the multiresolution analysis capability offered by the wavelet transform we can speed up the computation by ignoring the high-frequency details of the probability function up to a certain level.

Keywords: Wavelet transform, nonlinear filtering, conditional probability density function.

1 Introduction

The most fundamental approach of finding the value of the random state of stochastic dynamical system would be compute the conditional probability density functions (conditioned on all previous measurement data) of the state. The formulas for the conditional probability density function are quite simple and are derivable as an exercise of the Bayes' formula. The computational burden, however, of finding the entire probability density function numerically during every brief measurement time interval is formidable.

If the dynamical system is linear and the underlying processes are Gaussian then the state estimate remains to be a Gaussian random vector which is completely determined by its mean and covariance matrix only. Indeed the Kalman filter [1] computes the mean and the covariance matrix recursively for a linear system. Most real systems are however nonlinear, and a standard approach is to resort to the extended Kalman filter which employs certain linearization before applying the Kalman filter. Unfortunately, this linearization is not always satisfactory, and various direct nonlinear filtering algorithms have been proposed by many researchers [2], [3]. The trade-off between the computational speed and the accuracy has been always a major consideration in choosing a nonlinear filtering algorithm.

In this paper we consider an approach of representing and recursively generating approximation to the conditional probability density function using the wavelet transform [4]. The center idea is to recognize that the conditional probability density function waveform may be an excellent approximation to the original conditional probability density function.

The problem is defined in section 2 and the filtering algorithm as presented in section 3.

2 The nonlinear filtering problem

The basic problem addressed in this paper is to find the conditional probability density function of the state vector $x(k) \in \mathbf{R}^n$ which according to the discrete-time nonlinear system model:

$$x(k+1) = f(x(k), k) + u(k), \tag{1}$$

given the entire measurement history $Y_k = [y(1), y(2), ..., y(k)]$, where

$$y(k) = h(x(k)) + v(k), \tag{2}$$

where $y(k) \in \mathbf{R}^p$. Here, f and h are real, possibly nonlinear, vector-valued vector functions. The probability density function $p_{x(0)}(x)$ of the uncertain initial state $x(0)$ is assumed to be known. The process noise $u(k)$ and the measurement noise $v(k)$ are modeled as the independent random vectors with known probability density functions $p_{u(k)}(x)$ and $p_{v(k)}(x)$ respectively. Here $p_{x(0)}(x)$, $p_{u(k)}(x)$ and $p_{v(k)}(x)$ are not necessary Gaussian.

It is not difficult to show that the predicted conditional density function $p_{x(k+1)|Y_k}(x)$ and filtered conditional density function $p_{x(k+1)|Y_{k+1}}(x)$ can be obtained recursively starting from $p_{x(0)}(x) = p_{x(0)|y_0}(x)$ using

$$p_{x(k+1)|y_k}(x) = \int\limits_{-\infty}^{\infty} p_{u(k)}(x - f(y,k)) p_{x(k)|y_k}(y) dy, \tag{3}$$

$$p_{x(k+1)|y_{k+1}}(x) = \frac{1}{c} p_v(y(k+1) - h(x)) p_{x(k+1)|y_k}(x), \tag{4}$$

where c is the normalizing constant. Recursions 3 and 4 will be called the time- and the measurement-update, respectively.

3 Approximation of the time-update using the wavelet transform

With the simpler notations of $r(\cdot) = p_{u(k)}(\cdot)$ and $q(\cdot) = p_{x(k)|y_k}(\cdot)$ 3 can be rewritten as an n-dimensional convolution operation.

$$p_{x(k+1)|y_k}(x) = \int\limits_{-\infty}^{\infty} r(x-b)s(b)db, \tag{5}$$

where $b = f(y,k), f_1^{-1}(b) = y, s(b) = q(f_1^{-1}(b)) \left| det \frac{\partial y}{\partial b} \right|$, assuming that $f(\cdot, k)$ is a monotonic continuously differentiable function.

The convolution operation is the time-update 5 is computationally intensive. We shall show how to approximate the probability density functions $r(\cdot)$ and $s(\cdot)$ in 5 using the wavelet transformation to speed up the computation. We shall consider only the case when $x(k)$ in 1 is a scalar, i.e. $n = 1$ for the simplicity. The first step is to discretize the density function $r(\cdot)$ and $s(\cdot)$ $\{r_i\}_{i=0}^{N-1}$ and $\{s_i\}_{i=0}^{N-1}$. Then the continuous-domain convolution in 5 can be approximated with a discrete-domain convolution, i.e. $\sum\limits_{k=0}^{N-1} r_{i-k}s_k$.

The central limit theorem says that if a large number of functions are convolved together the resultant may be very smooth and as the number increases indefinitely the resultant may be approach to the Gaussian form. Recursion in 3 and 4 may be viewed as convolution operations and we may expect that the predicted conditional probability density function are are very smooth. Now a smooth function contains little high-frequency components and can be approximated with the approximation waveform at a higher level of the wavelet transform.

The operation $\sum\limits_{l=0}^{N-1} r_{i-l}s_l$ can be represented as the matrix-vector product $R \cdot g$, where

$$g = [g_0, g_1, ..., g_{N-1}]^T,$$

$$R = [r, Zr, Z^2 r, ..., Z^{N-1}r]^T,$$

$$r = [r_0, r_1, ..., r_{N-1}]^T,$$

$$Z = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & & \ddots & & \\ & & & & \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} - (N \times N - matrix).$$

Let $\{r_i^{(k)}\}_{i=0}^{N-1}, \{g_i^{(k)}\}_{i=0}^{N-1}, k = 0, 1, ...$ be the discrete sequence of the functions:

$$r_j^{(0)} = p_{u(0)}(x_j),\ g_j^{(0)} = p_{x(0)}(x_j),$$

$$r_j^{(1)} = p_{u(1)}(x_j),\ g_j^{(1)} = \sum_{l=0}^{N-1} r_{j-l}^{(1)} g_l^{(0)},$$

$$\dots\dots\dots\dots$$

$$r_j^{(k)} = p_{u(k)}(x_j),\ g_j^{(k)} = \sum_{l=0}^{N-1} r_{j-k}^{(1)} g_l^{(k-1)},$$

Wavelet-analysis begins with [5]

$$a_{0,m}^{(k)} := r_m^{(k)},\ \overline{a}_{0,m}^{(k)} := g_m^{(k)},\ m = \overline{0, N-1}, k = 0, 1, ...$$

$$a_{j,n}^{(k)} = \sum_l \overline{h_l} a_{j-1,2k+l}^{(k)},$$

$$d_{j,n}^{(k)} = \sum_l \overline{p_l} a_{j-1,2n+l},\ j = \overline{1, J},$$

where $\{\bar{h}_l\}$, $\{\bar{p}_l\}$ – the sequence, defined by the wavelet-function.

$\{d_{j,n}^{(k)}\}, j = J, J < N$, the coefficients of wavelet-decomposition of the $\{r_m^{(k)}\}$. The same formulas are obtained for $\{\bar{a}_{0,m}^{(k)}\} = g_m^{(k)}$.

The formula of synthesis is following

$$a_{j-1,n}^{(k)} = \sum_l h_{n-2l} a_{j,l}^{(k)} + \sum_l p_{n-2l} d_{j,l}^{(k)}. \tag{6}$$

For the concrete calculations we need the table of the values $\{h_l\}, \{p_l\}$.

4 Simulation results

We shall consider the following nonlinear system:

$$x(l+1) = 0.5x(l) + 2\cos(1.2l) + u(l),$$

$$y(l) = \frac{x^2(l)}{20} + v(l),$$

where $u(l) \sim \mathcal{N}(0; 0.5), v(l) \sim \mathcal{N}(0; 0.1)$, $\mathbf{E}\{x(0)\} = 10, p_{x(0)|y_0}(x) \sim 0.5\mathcal{N}(-4; 5) + 0.5\mathcal{N}(10; 2)$.

The table of the values $\{h_k\}, \{p_k\}$ corresponds to the Daubeshies wavelet, $\psi(x)$ [6].

k	0	1	2	3	4	5
h_k	0.33267	0.80689	0.45988	-0.13501	-0.08544	0.03522
$p_k = (-1)^k h_{5-k}$	0.03523	0.08544	-0.13501	0.45988	0.80689	-0.33267

The results of the simulation are following:

k	0	1	2	3	4	5	6	7	8	9	10
$x(k)$	12	9	0	-0.5	0	0.2	2	1.8	0.1	-2	-1
$\hat{x}(k)$	10	11	2	0	-2	0	1.8	1.7	0.1	-2.1	-1.2

This results shows the efficiency of the given approach.

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