Some model theory over an exceptional uniserial ring and decompositions of serial modules

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Abstract

We give an example of a direct summand of a serial module that does not admit an indecomposable decomposition.

1 Introduction

A module M is called *uniserial* if its lattice of submodules is a chain, and M is *serial* if it is a direct sum of uniserial modules. In recent years the theory of serial modules has been developed to a high level by many authors inspired by pioneering investigations by A. Facchini. The reader is referred to [5] for a general review and especially for a list of open problems.

Despite powerful ring and module theoretic methods used in the proofs, some core problems of this theory remain unsolved. For instance it has not been known up til now whether every direct summand of a serial module is serial. This problem seems to be similar to the problem of Matlis of whether a direct summand of a direct sum of indecomposable injective modules admits a similar decomposition.

In this paper we give a negative answer to the former question. Precisely we construct a direct summand M of a serial module such that M can not be represented as a direct sum of indecomposable modules, in particular M is not serial.

This example is found among the pure projective modules over a very special uniserial ring — a so-called exceptional uniserial ring. A uniserial ring R is said to be *nearly simple* if R is not artinian and the unique nontrivial

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twosided ideal of R is its Jacobson radical. Puninski [11] has classified the pure projective modules over nearly simple uniserial domains. It turned out that over such a domain there are only three indecomposable pure projective modules (all are uniserial) and every pure projective module is a direct sum of indecomposables. But there is a small difference compound with the usual property called finite representation type. Though finite indecomposable decompositions of pure projectives over R behave very well, there are some nontrivial identifications between pure projective modules of infinite Goldie dimension.

A nearly simple uniserial ring R is called *exceptional* if R is prime and contains zero divisors. In fact even the existence of such a ring has been considered as a difficult problem and the first example was found by Dubrovin [2]. Continuing the lines of [11], in this paper we investigate pure projective modules over an exceptional uniserial ring R. The above-mentioned three uniserial pure projective modules also appear here. But one needs an additional pure projective module W to state (conjecturally) that every pure projective module over R is a direct sum of these four, and we prove that W does not admit an indecomposable decomposition. Since every pure projective module over a uniserial ring is a direct summand of a serial module, we obtain the desired counterexample.

The basis for almost all considerations in this paper is a precise description of the lattice of 1-pp-formulae over an exceptional uniserial ring. So many proofs involve as a searching for an appropriate decomposition in this lattice. The machinery used is a mixture of general ring theory and model theory of modules, where we try to combine the advantages of each approach. Roughly speaking, ring theory is good for obtaining rough global results, whereas the model theory of modules works on the level of a concrete module.

For instance we prefer to consider countably generated pure projective modules over a ring in the spirit of Ph. Rothmaler [7] as pp-atomic (informally "small") modules. That makes it advantageous to bypass problems in proving that a fixed module M is pure projective by considering M locally.

2 Preliminaries

Let M be a right module and $N \subset M$, then $\mathrm{ann}_R(N) = \{r \in R \mid nr = 0$ for every $n \in N\}$ is a right ideal of R. Similarly for $S \subset R$, an *annihilator subgroup* of M, written $\mathrm{ann}_M(S)$, is $\{m \in M \mid ms = 0$ for every $s \in S\}$.

Also $\operatorname{lann}_R(S) = \{r \in R \mid rS = 0\}$ is a left ideal of R and $\operatorname{rann}_R(S) = \{r \in R \mid Sr = 0\}$ is a right ideal of R.

Given a right ideal I of R let us define the *second annihilator* of I, written $\operatorname{ann}_2(I)$, as $\operatorname{rann}_R[\operatorname{lann}_R(I)]$. Clearly $I \subseteq \operatorname{ann}_2(I)$. Similarly for a left ideal J of R, the left ideal $\operatorname{ann}_2(J)$ is $\operatorname{lann}_R[\operatorname{rann}_R(J)]$.

A module M is *locally coherent* if every finitely generated submodule of M is finitely presented. By [14, 26.1] any factor of a locally coherent module by a finitely generated submodule is locally coherent. Also an arbitrary direct sum of locally coherent modules is locally coherent. A module M is said to be *coherent* if M is finitely generated and locally coherent. A ring R is *right (left) coherent* if the module R_R ($_RR$) is coherent and R is *coherent* means R is both left and right coherent. R is right coherent if and only if the intersection of every pair of finitely generated right ideals of R is a finitely generated ideal and, moreover, every right ideal $\operatorname{rann}_R(r)$, $r \in R$, is finitely generated.

A module M is said to be *uniserial* if the lattice of submodules of M is a chain and M is *serial* if M is a direct sum of uniserial modules. A ring R is right (left) *uniserial* if the module R_R ($_RR$) is uniserial. R is *uniserial* if it is right uniserial and left uniserial. $\operatorname{Jac}(R)$ will denote the Jacobson radical of ring R. Since every uniserial ring R is local, $\operatorname{Jac}(R)$ is the largest right (left, twosided) ideal of R. Also $F = R/\operatorname{Jac}(R)$ is a skew field.

Remark 2.1 *A uniserial ring R is right coherent iff for all (any) $0 \neq r \in \operatorname{Jac}(R)$, $\operatorname{rann}_R(r)$ is a principal right ideal.*

Proof. Since every finitely generated right ideal of a uniserial ring is principal, an intersection of two finitely generated right ideals is a principal right ideal. So for right coherence it suffices to check that $\operatorname{rann}_R(s)$ is a principal right ideal for every $0 \neq s \in \operatorname{Jac}(R)$. Let us assume that $\operatorname{rann}_R(r) = tR$. If $r = sv$ then clearly $\operatorname{rann}_R(s) = v \cdot \operatorname{rann}_R(r) = vtR$. Otherwise $s = rw$, hence writing $t = wt'$ we obtain $\operatorname{rann}_R(s) = t'R$. □

Over a uniserial ring R by [5, Theorem 9.19] every finitely presented module is a finite direct sum of uniserial modules R/r_iR, $r_i \in R$ and this decomposition is unique.

We say that a uniserial ring R is *nearly simple* if $\operatorname{Jac}(R)$ is the unique nontrivial twosided ideal of R and R is not artinian. Equivalently, a uniserial ring R is nearly simple if and only if $\operatorname{Jac}(R)$ is the unique nontrivial twosided ideal of R and $\operatorname{Jac}^2(R) \neq 0$. A uniserial ring R is *exceptional* if R is nearly simple, prime and contains zero divisors. If R is a uniserial ring then the set of right (left) zero divisors coincides with right (left) singular ideal of R, in particular it is a twosided ideal of R.

A module M_R is said to be *right p-injective* if M is injective with respect to the embeddings $rR \subseteq R_R$, $r \in R$. Also M_R is *right fp-injective* if M is injective with respect to the embeddings of right modules $F \subseteq R^n$, where F is finitely generated. The ring R is *right p-injective* (*right fp-injective*) if R_R is p-injective (fp-injective). Over a uniserial ring every p-injective module is fp-injective.

Lemma 2.2 *Let R be an exceptional uniserial ring. Then every $r \in \mathrm{Jac}(R)$ is a right and left zero divisor. Thus R is left and right fp–injective, in particular* $\mathrm{ann}_2(rR) = rR$ *and* $\mathrm{ann}_2(Rr) = Rr$ *for every $r \in R$. If additionally R is right coherent then it is left coherent.*

Proof. Since R contains zero divisors, the right singular ideal of R coincides with $\mathrm{Jac}(R)$, and similarly on the left. Since $\mathrm{Jac}(R)$ is the right and left singular ideal of R, it follows that R is right and left fp–injective by [13, Theorem 2.1]. The final equalities hold in every p-injective ring by [8, Lemma 1.1].

For left coherence let $0 \neq s \in \mathrm{Jac}(R)$ and $\mathrm{rann}_R(s) = tR$. Then $\mathrm{lann}_R(t) = Rs$ by the double annihilator condition, hence R is left coherent by Remark 2.1. $\square$

Now we recall some notions from the model theory of modules. For more information the reader is referred to M. Prest's book [9]. All facts from the model theory of modules over a uniserial ring cited below can be found in [4] and [12].

A pp-formula (in one free variable) $\varphi(x)$ over a ring R is a formula $\exists\, \bar{y} = (y_1, \ldots, y_n)\, \bar{y}A = x(b_1, \ldots, b_k)$ where A is an $n \times k$ matrix over R and $b_i \in R$. For an element m of a module M we say that φ is *satisfied by m in M*, written $M \models \varphi(m)$, if there is a tuple $\overline{m} = (m_1, \ldots, m_n)$ in M such that $\overline{m}A = m(b_1, \ldots, b_k)$. Clearly $\varphi(M) = \{m \in M \mid M \models \varphi(m)\}$ is a subgroup in M called a *pp-subgroup*, and $\varphi(M)$ is even a left S-submodule of M for $S = \mathrm{End}(M)$.

A *divisibility formula* is a pp-formula $a \mid x$, where $a \in R$ and an *annihilator formula* is a pp-formula $xb = 0$, where $b \in R$. For instance $(a \mid x)(M) = Ma$ and $(xb = 0)(M) = \mathrm{ann}_M(b)$ for every module M. Over a uniserial ring divisibility formulae form a chain and the same is true for annihilator formulae. Also these chains generate the lattice of all pp-formulae over a uniserial ring, in particular this lattice is distributive. So every pp-formula $\varphi(x)$ over a uniserial ring R is equivalent to a finite conjunction of pp-formulae $ab \mid xb$, $a, b \in R$ and also to a finite sum of pp-formulae $c \mid x \wedge xd = 0$, $c, d \in R$.

A pair (M, m) where M is a module and $m \in M$ is called a *free realization* of a pp-formula $\varphi(x)$, if $M \models \varphi(m)$ and for every pp-formula $\psi(x)$, $M \models \psi(m)$ implies $\varphi \to \psi$ where the latter means that $\varphi(N) \subseteq \psi(N)$ for every module N. For instance the pair $(R/bR, 1)$ is a free realization for $xb = 0$, and $(R/abR, a)$, $ab \neq 0$ is a free realization for $a \mid x \wedge xb = 0$.

An inclusion of modules $M \subseteq N$ is said to be *pure* if $\varphi(M) = M \cap \varphi(N)$ holds for every pp-formula $\varphi(x)$. Over a uniserial ring for purity it suffices to verify that $Mr = M \cap Nr$ for every $r \in R$. A module M is *pure injective* if it is injective over pure embeddings. Over a uniserial ring every indecomposable pure injective module M is uniserial as a module over its endomorphism ring. In particular the lattice of pp-subgroups of M is a chain, so M is *pp-uniserial*.

For every module M there exists a "minimal" pure injective module $\mathrm{PE}(M)$ that contains M as a pure submodule. Over a uniserial ring a module $\mathrm{PE}(M)$ is indecomposable if and only if M is pp-uniserial and connected. The latter means that for every $0 \neq m, n \in M$ there is a pp-formula $\varphi(x, y)$ such that $M \models \varphi(m, n) \wedge \neg\, \varphi(m, 0)$.

When we refer to a *pair* of pp-formulae (φ/ψ) we usually mean that $\psi \to \varphi$, otherwise we can replace it by the pair $(\varphi/\varphi \wedge \psi)$. If $\psi(M) \subset \varphi(M)$ for a module M we say that M *opens the pair* (φ/ψ). The underlying set for the Ziegler spectrum Zg_R over a ring R is the set of isomorphism types of indecomposable pure injective R-modules. A pair (φ/ψ) can be also interpreted as the open set $\{M \in \mathrm{Zg}_R \mid M \text{ opens } (\varphi/\psi)\}$ and all such sets form a basis for Zg_R. Every basic open set in Zg_R is quasi–compact.

If m is an element of a module M then the collection of pp-formulae $pp_M(m) = \{\varphi(x) \mid M \models \varphi(m)\}$ is called a *pp-type*. For every pp-type p there exists a "minimal" pure injective module $N(p)$ realizing p. For instance if M is an indecomposable pure injective module, $0 \neq m \in M$ and $p = pp_M(m)$, then $N(p) \cong M$. A pp-type $p(x)$ is called *indecomposable* if the module $N(p)$ is indecomposable. Over a uniserial ring R a pp-type p is indecomposable if and only if for every $a, b \in R$, $ab \mid xb \in p$ implies either $a \mid x \in p$ or $xb = 0 \in p$.

An epimorphism of modules $f : M \to N$ is *pure* if $\varphi(M) = f^{-1}(\varphi(N))$ holds for every pp-formula $\varphi(x)$. A module M is *pure projective* if M is projective with respect to pure epimorphisms. Over an arbitrary ring a module is pure projective if and only if it is a direct summand of a direct sum of finitely presented modules. So a module M over a uniserial ring R is pure projective if and only if M is a direct summand of a module $\oplus_i R/r_iR$, $r_i \in R$. Over any ring a factor of a pure projective module by a finitely generated submodule is pure projective. Also every finite tuple of elements of a pure projective module M realizes in M a pp-type p generated by a single pp-formula (we say p is *finitely generated*).

The following fact shows that the converse is also true, at least for countably generated modules.

Fact 2.3 *[7, Theorem 3.1 + Proposition 2.10] Let M be a countably generated module such that the pp-type of every finite tuple $\overline{m} \in M$ is finitely generated. Then M is pure projective.*

3 Pure injective modules

It is well known (see [6, p. 535]) that for a ring R the module $\mathrm{PE}(R_R)$ is indecomposable if and only if R is local. For instance for every uniserial ring R the module $\mathrm{PE}(R_R)$ is indecomposable. There is another way to express this in our particular situation.

Corollary 3.1 *Let R be an exceptional uniserial ring. Then $\mathrm{PE}(R_R)$ is an indecomposable injective module.*

Proof. By Lemma 2.2, the module R_R is fp–injective. Since $\mathrm{PE}(R_R) = N(p)$ for $p = pp_R(1)$, we have $\mathrm{PE}(R_R) = \mathrm{E}(R_R)$. Since R_R is a uniserial module, $\mathrm{E}(R_R)$ is indecomposable. □

There is a counterpart to this result.

Lemma 3.2 *Let R be an exceptional uniserial ring and $0 \neq r \in \mathrm{Jac}(R)$. Then $\mathrm{PE}(rR)$ is a decomposable module.*

Proof. It suffices to check that rR is not pp–uniserial. By Lemma 2.2, $rt = 0$ for some $0 \neq t \in R$. Since R is prime, $r\,\mathrm{Jac}(R)t \neq 0$, and hence $rst \neq 0$ for some $0 \neq s \in \mathrm{Jac}(R)$. Then $rs \in (s \mid x)(rR)$ and $rs \notin (xt = 0)(rR)$. Also clearly $r \in (xt = 0)(rR)$ and $r \notin (s \mid x)(rR)$. Thus the pp-subgroups $(s \mid x)(rR)$ and $(xt = 0)(rR)$ are incomparable. □

In the following claim we use the fact that every indecomposable pure injective module over a uniserial ring is pp-uniserial.

Lemma 3.3 *Let R be an exceptional uniserial ring, $0 \neq a \in \mathrm{Jac}(R)$ and $b, c \in R$ such that $cb \neq 0$. Then $a \mid x \wedge xb = 0 \to c \mid x$.*

Proof. Assuming the contrary we obtain that the pair $(a \mid x \wedge xb = 0/c \mid x)$ is open on some indecomposable pure injective R-module M. If M is not faithful then $M \cong F = R/\operatorname{Jac}(R)$, hence $Ma = 0$, a contradiction. Thus M is faithful, so $Mcb \neq 0$ implies that Mc is not a subset of $\operatorname{ann}_M(b)$. Since M is pp-uniserial, $\operatorname{ann}_M(b) \subset Mc$, also a contradiction. □

There is an important consequence of this result.

Lemma 3.4 *Let R be an exceptional uniserial ring. Then $\operatorname{PE}(\operatorname{Jac}(R)_R)$ is indecomposable and not fp-injective.*

Proof. Again it suffices to prove that $\operatorname{Jac}(R)$ is pp-uniserial. We know that the lattice of all pp-formulae over a uniserial ring is generated by divisibility formulae $a \mid x$, $a \in R$ and annihilator formulae $xb = 0$, $b \in R$. Also clearly $a \mid x \to xb = 0$ if and only if $ab = 0$.

Thus it is enough to prove that $(xb = 0)(\operatorname{Jac}(R)) \subseteq \operatorname{Jac}(R)a$ for every $0 \neq a, b \in \operatorname{Jac}(R)$ such that $ab \neq 0$. Let $0 \neq t \in \operatorname{Jac}(R)$ be such that $tb = 0$. Since $\operatorname{Jac}(R)$ is not principal as a left ideal, $t = t'j$ for some $0 \neq t', j \in \operatorname{Jac}(R)$. Thus the formula $j \mid x \wedge xb = 0$ is satisfied by t in $\operatorname{Jac}(R)$. Since $ab \neq 0$, Lemma 3.3 yields that $a \mid x$ is true on t in $\operatorname{Jac}(R)$, hence $t \in \operatorname{Jac}(R)a$.

Thus $\operatorname{PE}(\operatorname{Jac}(R)_R)$ is an indecomposable module. Also for arbitrary $0 \neq j \in \operatorname{Jac}(R)$ we have that $j \notin \operatorname{Jac}(R)j$, but $j \in Rj$, hence $\operatorname{Jac}(R)$ is not fp–injective. □

Note that all nontrivial indecomposable finitely presented modules over an exceptional uniserial ring are isomorphic.

Remark 3.5 *Let R be an exceptional uniserial ring. Then all modules of the form R/rR, $0 \neq r \in \operatorname{Jac}(R)$ are isomorphic.*

Proof. Since R is nearly simple, the result follows by [11, Corollary 4.3]. □

The following is a description of the indecomposable pure injective modules over an exceptional uniserial ring.

Proposition 3.6 *Let M be an indecomposable pure injective module over an exceptional uniserial ring R. Then exactly one of the following is the case:*

1) M is indecomposable injective, hence $M \cong \operatorname{E}(R/I)$ for some right ideal I of R;

2) $M \cong \operatorname{PE}((\operatorname{Jac}(R))_R)$ is not fp-injective;

3) $M \cong F = R/\operatorname{Jac}(R)$ is Σ-pure injective.

Proof. If M is not faithful, then, since R is nearly simple, M is an $F = R/\operatorname{Jac}(R)$-module, hence $M \cong F$. Clearly M is Σ-pure injective over F, hence over R.

Otherwise M is faithful. If M is injective, then clearly $M \cong \mathrm{E}(R/I)$ for some right ideal I of R. Also since R/I is a uniserial module, its injective envelope is indecomposable.

Thus we may assume that M is faithful and not injective. Let $0 \neq m \in M$, $p = pp_M(m)$, $I = \operatorname{ann}_R(m)$ and $J = \{s \in R \mid m \notin Ms\}$. Then I is a right ideal and J is a left ideal of R.

By the representation theorem (see [12, Theorem 17.17]), p is realized as the pp-type of some element t in a (faithful) right ideal T of R (in fact since R is prime, every nonzero right ideal of R is faithful), in particular $I \neq 0$. If $T = tR$, then by Lemma 3.2, p is decomposable, a contradiction. Thus $tR \subset T$, hence $g \mid x \in p$ for some $0 \neq g \in \operatorname{Jac}(R)$. Thus for every $i \in I$ we have $g \mid x \wedge xi = 0 \in p$.

Let $q = pp_{E(T)}(t)$, hence $p \subseteq q$. If $p = q$ then M is a direct summand of $\mathrm{E}(T)$, hence injective, a contradiction. Otherwise $a \mid x \in q \setminus p$ for some $a \in R$. (We use here that if p is indecomposable then $ab \mid xb \in p$ implies either $a \mid x \in p$ or $xb = 0 \in p$). Since $a \mid x \in q$, we have $\operatorname{rann}_R(a) \subseteq I$. Also since $a \mid x \notin p$, Lemma 3.3 yields $ai = 0$ for every $i \in I$, hence $aI = 0$. Thus $\operatorname{rann}_R(a) = I$.

We prove that $J = Ra$. Clearly $a \in J$. If $J \neq Ra$ we get $a' \in J$ for some $a' \in R$ such that $a \in \operatorname{Jac}(R)a'$. Then $\operatorname{rann}_R(a') \subseteq I$ hence as above we obtain $\operatorname{rann}_R(a') = I$. Then Lemma 2.2 implies $Ra = Ra'$, a contradiction.

Thus $\operatorname{rann}_R(a) = I$ and $Ra = \{s \in R \mid a \notin \operatorname{Jac}(R)s\} = J$. Moreover the pp-type $r = pp_{\operatorname{Jac}(R)}(a)$ is indecomposable by Lemma 3.4. Thus p coincides with r on divisibility and annihilator formulae. Since p is indecomposable, $p = r$. Also the module $\mathrm{PE}(\operatorname{Jac}(R))$ is indecomposable, therefore $M \cong \mathrm{PE}(\operatorname{Jac}(R))$. $\square$

4 The lattice of pp-formulae

We remark that for $a, b \in R$ with $ab = 0$ we obviously have $a \mid x \to xb = 0$, hence the formula $a \mid x \wedge xb = 0$ is equivalent to the formula $a \mid x$. We will treat this case as *trivial*. The following claims will show that each nontrivial formula $a \mid x \wedge xb = 0$ defines a proper section on the chain of divisibily formulae and also all such formulae are ordered into a chain by annihilator conditions.

Lemma 4.1 *Let $a, b \in \mathrm{Jac}(R)$ be such that $ab \neq 0$ and $c \in R$. Then $c \mid x \to a \mid x \wedge xb = 0$ iff $cb = 0$ and then this implication is proper. Otherwise $a \mid x \wedge xb = 0 \to c \mid x$ and that is also a proper implication.*

Proof. Since $ab \neq 0$, we have $Ra \cap \mathrm{lann}_R(b) = \mathrm{lann}_R(b)$.

So, if $cb = 0$, then $c \in Ra$. Thus $c \mid x \to xb = 0$ and $c \mid x \to a \mid x$, hence $c \mid x \to a \mid x \wedge xb = 0$.

If $cb \neq 0$, then $a \mid x \wedge xb = 0 \to c \mid x$ by Lemma 3.3.

Now the conclusion is evident. □

Lemma 4.2 *Let $a, b, a', b' \in \mathrm{Jac}(R)$ where $ab, a'b' \neq 0$. Then $a \mid x \wedge xb = 0 \to a' \mid x \wedge xb' = 0$ iff $b' \in bR$. For instance if $b' \in b\,\mathrm{Jac}(R)$ then this implication is proper.*

Proof. Let us assume that $b' \in bR$. Then $xb = 0 \to xb' = 0$, hence $a \mid x \wedge xb = 0 \to xb' = 0$. If $a \mid x \wedge xb = 0 \to a' \mid x$ were not true, then Lemma 4.1 would give $a'b = 0$, hence $a'b' = 0$, a contradiction. Thus $a \mid x \wedge xb = 0 \to a' \mid x$ which clearly yields $a \mid x \wedge xb = 0 \to a' \mid x \wedge xb' = 0$.

Also suppose that $b' \in b\,\mathrm{Jac}(R)$ and $a' \mid x \wedge xb' = 0 \to a \mid x \wedge xb = 0$. Then $a' \mid x \wedge xb' = 0 \to xb = 0$. But in the module $(R/a'b'R, a')$ we clearly have $a'b \neq 0$, a contradiction. □

Now by elementary duality we immediately obtain that the formula $xa = 0 + b \mid x$, $ba \neq 0$ defines a proper section on the chain of annihilator formulae and all such formulae are linearly ordered by the divisibility condition.

Corollary 4.3 *Let $a, b \in \mathrm{Jac}(R)$ be such that $ba \neq 0$. Then $xc = 0 \to xa = 0 + b \mid x$ iff $bc \neq 0$ and then this implication is proper. Otherwise $xa = 0 + b \mid x \to xc = 0$ and then that is also a proper implication. Also if $a', b' \in \mathrm{Jac}(R)$ with $b'a' \neq 0$ then $(xa = 0 + b \mid x) \to (xa' = 0 + b' \mid x)$ iff $b \in Rb'$ and this implication is proper if $b \in \mathrm{Jac}(R)b'$.*

This leads to a complete description of lattice of all pp-formulae over an exceptional uniserial ring.

Proposition 4.4 *Let R be an exceptional uniserial ring. Then the lattice of all pp-formulae over R is as shown in Figure 1, where $0 \neq a, b, c, d \in \mathrm{Jac}(R)$, $ad = 0$, $c \in Ra$, $cb \neq 0$.*

Proof. By Lemma 3.3 and Corollary 4.3, we have two chains in the lattice of all pp-formulae over R that include the set of generators for this lattice.

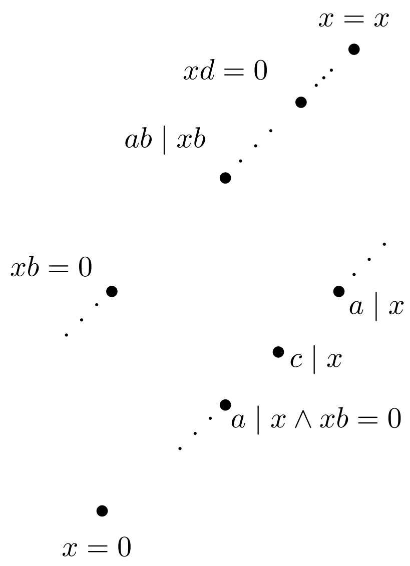

Figure 1:

So it suffices to show that the sum and the intersection of elements from different chains lie in one of those. By elementary duality and symmetry it is enough to consider the case $(a \mid x \wedge xb = 0) + xb' = 0$ for $a, b, b' \in \mathrm{Jac}(R)$, $ab \neq 0$, $b' \notin bR$. By modularity we transform this formula to $(a \mid x + xb' = 0) \wedge xb = 0$. Since $ab \neq 0$ by Corollary 4.3 this formula is equivalent to $xb = 0$. $\square$

Over an exceptional uniserial coherent ring flatness is the same as fp-injectivity.

Corollary 4.5 *Over an exceptional uniserial coherent ring R every fp-injective module is flat and every flat module is fp-injective. Also* $\mathrm{Jac}(R)$ *is neither flat nor fp-injective.*

Proof. Every fp-injective R-module is flat and every flat R-module is fp-injective by [14, 48.8].

Since $\mathrm{Jac}(R)_R$ is not fp-injective, it is not flat. $\square$

Now we investigate isolated points in the Ziegler spectrum over an exceptional uniserial coherent ring.

Lemma 4.6 *Let R be an exceptional uniserial coherent ring. Then F and* $\mathrm{PE}(\mathrm{Jac}(R_R))$ *are the only isolated points in* Zg_R. *Also* Zg'_R *consists only*

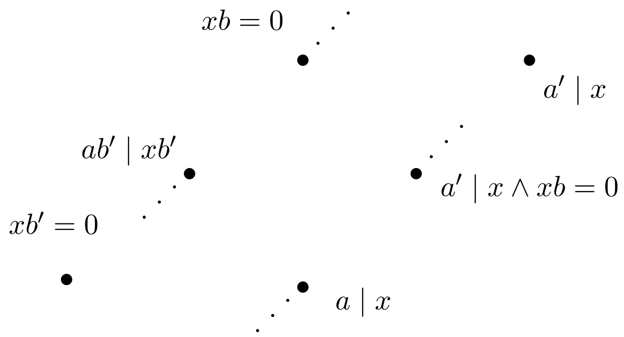

Figure 2:

of injective flat points and contains no isolated point. Thus the Cantor–Bendixson rank of Zg_R *is undefined.*

Proof. Let $0 \neq a \in \mathrm{Jac}(R)$, $\mathrm{rann}_R(a) = bR$ and let $a', b' \in \mathrm{Jac}(R)$ be such that $b \in b' \,\mathrm{Jac}(R)$, $a \in \mathrm{Jac}(R)a'$. Then (see Figure 2) we obtain a quadruple in the lattice of pp-formulae over R each side of which is a minimal pair of pp-formulae. In particular a pair $(xb = 0/a' \mid x \wedge xb = 0)$ isolates F as does the perspective minimal pair $(ab' \mid xb'/a \mid x)$. The pair $(xb = 0/ab' \mid xb')$ and also the pair $(a' \mid x \wedge xb = 0/a \mid x)$ isolate $\mathrm{PE}(\mathrm{Jac}(R))$.

Suppose that there exists a point in Zg'_R that is isolated by a pair (φ/ψ). In any injective module M by [10, Proposition 1.3] we have $\varphi(M) = \mathrm{ann}_M(D\varphi({}_RR))$ where we use D for the dual pp-formula. Since R is coherent, $D\varphi({}_RR) = aR$ for some $a \in R$, thus φ is equivalent in the theory of injective modules to $xa = 0$. Similary ψ is equivalent to $xb = 0$, hence $0 \neq a = bc$ for $c \in \mathrm{Jac}(R)$.

But both modules $(\mathrm{E}(R/aR), 1)$ and $(\mathrm{E}(R/b\,\mathrm{Jac}(R)), 1)$ open this pair. Clearly they are nonisomorphic, a contradiction. □

As we have seen during this proof, the lattice of pp-formulae over an exceptional uniserial coherent ring consists of quadruples that form a dense linear order.

Now we consider the pure injective envelope of a module R/rR, $0 \neq r \in \mathrm{Jac}(R)$. If $\mathrm{lann}_R(r) = Rs$ then $sR \cong R/rR$, hence by Lemma 3.2, $\mathrm{PE}(R/rR)$ is a decomposable module. We point out the exact form of this decomposition.

Lemma 4.7 *Let R be an exceptional uniserial coherent ring and let $0 \neq r \in \mathrm{Jac}(R)$. Then $\mathrm{PE}(R/rR) \cong \mathrm{PE}(\mathrm{Jac}(R)_R) \oplus F$.*

Proof. Let $\mathrm{lann}_R(r) = Rs$ for $s \in R$, hence $\mathrm{rann}_R(s) = rR$. We define an embedding $f : M = R/rR \to \mathrm{Jac}(R) \oplus F = N$ by the rule $f(1) \to (s, 1)$. Let us verify that this embedding is pure, where it suffices to check only the divisibility conditions. Clearly for every $0 \neq s \in \mathrm{Jac}(R)$ we have $1 \notin Ms$ and also $f(1) = (s, 1) \notin Ns = (\mathrm{Jac}(R)s, 0)$.

So let $m \in M$ be the image of some $0 \neq j \in \mathrm{Jac}(R)$ such that t divides $f(j) = (sj, 0)$ in N, hence t divides sj in $\mathrm{Jac}(R)$. Then $t \notin Rsj$. We write $r = jh$ for $0 \neq h \in \mathrm{Jac}(R)$. Thus the formula $j \mid x \wedge xh = 0$ is satisfied by m in M. Letting $m \notin Mt$, by Lemma 4.1 we get $th = 0$. But $0 = sr = sjh$ clearly implies $\mathrm{lann}_R(h) = Rsj$, hence $t \in Rsj$, a contradiction. □

Corollary 4.8 *Let R be an exceptional uniserial coherent ring, $0 \neq r \in \mathrm{Jac}(R)$ and $S = \mathrm{End}(R/rR)$. Then S has exactly two maximal right (left, twosided) ideals: K consisting of non epimorphisms and L consisting of non monomorphisms.*

Proof. Since R/rR is finitely presented and $\mathrm{PE}(R/rR)$ is decomposable, S is not a local ring. Thus by [5, Theorem 9.1] S has exactly two maximal right (left, twosided) ideals of the prescribed form. □

5 Pure projective modules

In this section we begin an investigation of pure projective modules over an exceptional uniserial ring R. Everywhere in the sequel we will assume that R is an exceptional uniserial coherent ring. The reasons for such assumption are twofold. The first is that the unique example of an exceptional uniserial ring known to us is Dubrovin's example and this is coherent (see proof below). The other is that if an exceptional uniserial noncoherent ring were to exist, the theory of pure projectives over it would look very similar to that over a nearly simple uniserial domain.

Lemma 5.1 *Let R be an exceptional uniserial ring constructed in [2, Theorem 2.5]. Then R is coherent.*

Proof. A basis for Dubrovin's example is given by a special right ordering on a trefoil group G to which he associated a uniserial domain R' (see [2] for precise definitions). One takes g in the center of G, hence the right ideal $I = gR'$ is two sided and one puts $R = R'/I$. By Remark 2.1 for coherence it suffices to check that $\mathrm{rann}_R(r)$ is principal for some $0 \neq r \in \mathrm{Jac}(R)$. Clearly we can take $r = g' \in G$, hence $\mathrm{rann}_R(r) = (g'^{-1}g)R$. $\square$

Let us make first a quite trivial remark.

Remark 5.2 *For every uniserial ring R and $0 \neq r \in \mathrm{Jac}(R)$, the module R/rR is coherent. Hence every pure projective module over an exceptional uniserial coherent ring is locally coherent.*

Proof. Indeed it suffices to prove that every nonzero finitely generated submodule N of $M = R/rR$ is finitely presented. Since N is finitely generated it is cyclic, say $N = sR$ for $0 \neq s \in R$. Then $r = st$ for $0 \neq t \in R$. It is evident that $N \cong R/tR$.

If R is an exceptional uniserial coherent ring, then every pure projective R-module is locally coherent, being a direct summand of the locally coherent module $R^{(\alpha)} \oplus (R/rR)^{(\beta)}$. $\square$

Recall that for every $0 \neq r \in \mathrm{Jac}(R)$ the endomorphism ring $S = \mathrm{End}(R/rR)$ contains exactly two maximal right (left, twosided) ideals K and L. Then $\mathrm{Jac}(S) = K \cap L$ and $S/\mathrm{Jac}(S)$ is a sum of two skew fields $S/K \oplus S/L$. The following is the first structure result for pure projectives over R.

Lemma 5.3 *Let M be the pure projective module over an exceptional uniserial coherent ring R. Then there is a decomposition $M = T \oplus U$ such that either 1) $T \cong R^{(\alpha)}$ and U has no direct summand isomorphic to R_R or 2) $T \cong (R/rR)^{(\beta)}$ for some (any) $0 \neq r \in \mathrm{Jac}(R)$ and U has no direct summand isomorphic to R/rR.*

Proof. Let $N = R \oplus (R/rR)$ and set $S' = \mathrm{End}(N)$. Clearly

$$S' = \begin{pmatrix} R & \mathrm{Hom}(R/rR, R) \\ R/rR & \mathrm{End}(R/rR) \end{pmatrix},$$

where we use the identification $\mathrm{Hom}(R, R/rR) = R/rR$. Since every $f \in \mathrm{Hom}(R, R/rR)$ is not mono and every $g \in \mathrm{Hom}(R/rR, R)$ is not epi, therefore $e_1Se_2Se_1 \subseteq \mathrm{Jac}(R)$ and $e_2Se_1Se_2 \subseteq K \cap L = \mathrm{Jac}(S)$. Thus the maximal right (left, twosided) ideals of S' are the following:

$$J' = \begin{pmatrix} \mathrm{Jac}(R) & \mathrm{Hom}(R/rR, R) \\ R/rR & \mathrm{End}(R/rR) \end{pmatrix} \quad K' = \begin{pmatrix} R & \mathrm{Hom}(R/rR, R) \\ R/rR & K \end{pmatrix}$$

and

$$L' = \begin{pmatrix} R & \mathrm{Hom}(R/rR, R) \\ R/rR & L \end{pmatrix}.$$

Since N is finitely generated, by [5, Proposition 3.12], $\mathrm{Hom}(N, -)$ defines a 1-1 correspondence between direct summands of $N^{(\gamma)}$ (hence pure projective right R-modules) and projective right modules over S'. Via this correspondence R goes to e_1S' and R/rR to e_2S'. Let P be a projective S'-module that corresponds to a pure projective R-module M. By Kaplansky's theorem we may assume that P is countably generated.

As in [3, Proposition 2.9] if P/PJ', P/PK' and P/PL' are infinite dimensional vector spaces (over the corresponding skew fields S'/J', S'/K' and S'/L'), then by Bass's theorem P is free, hence $M = T \oplus U$ where $T \cong R^{(\alpha)}$ and $U \cong (R/rR)^{(\beta)}$.

Also if P/PJ' is finite dimensional, we can split off finitely many copies of R in M, hence $M = R^n \oplus U$ where U contains no copies of R as a direct summand. Similarly if one of P/PK' and P/PL' is finite dimensional, then there is a decomposition $M = (R/rR)^m \oplus U$ where U contains no copies of R/rR as a direct summand (see [3, proof of prop. 2.9] for all of this). □

It is evident that the pp-formula $a \mid x$ has (R, a) as a free realization. Hence if a module M has R_R as a direct summand, the pp-type $pp_M(a)$ is generated by $a \mid x$. The converse is also true in our case, as the next lemma shows.

Lemma 5.4 *Let M be a pure projective module over an exceptional uniserial ring R. Let $m, n \in M$ be such that $na = m$, $0 \neq a \in R$ and the pp-type $p = pp_M(m)$ is generated by $a \mid x$. Then $nR \cong R_R$ is a direct summand of M.*

Proof. Clearly that $nR \cong R_R$ via $n \to 1$. Since R is fp-injective, nR is pure in M. Thus nR is a direct summand of M, being a pure finitely generated submodule of a pure projective module. □

Almost the same is true for a realization of a formula $ab \mid xb = a \mid x + xb = 0$.

Lemma 5.5 *Let M be a pure projective module over an exceptional uniserial ring R, let $0 \neq m \in M$ and set $p = pp_M(m)$. If p is generated by either a pp-formula $a \mid x$ for some $0 \neq a \in R$ or by $ab \mid xb$ for some $a, b \in \mathrm{Jac}(R)$ with $ab \neq 0$, then M has a direct summand isomorphic to R.*

Proof. If p is generated by $a \mid x$, the result follows by Lemma 5.4.

Let us assume that p is generated by $ab \mid xb$. In particular (since $ab \mid xb$ implies neither $a \mid x$ nor $xb = 0$), $m \notin Ma$ and $mb \neq 0$. Let $n \in M$ be such that $nab = mb$, hence $(m - na)b = 0$ and $m = (m - na) + na$. We assert that the pp-type $q = pp_M(na)$ is generated by $a \mid x$. Indeed clearly $a \mid x \in q$. Let $\varphi \in q$ for some pp-formula φ that is not implied by $a \mid x$. Then (see Figure 1) $\varphi + (xb = 0)$ is strictly less than $ab \mid xb$, a contradiction. So $a \mid x$ generates q in this case, hence M contains R as a direct summand by Lemma 5.4 again. $\square$

Now we find some examples of pure projective modules.

Lemma 5.6 *Let M be a countably generated locally coherent module over an exceptional uniserial coherent ring such that R_R is not a (pure) submodule of M. Then M is pure projective.*

Proof. By Fact 2.3, it suffices to verify that every finite tuple from M realizes a finitely generated pp-type. Let N be the submodule of M generated by this tuple. Choose a new system of generators $m_1, \ldots, m_k$ for N such that the number of $m_i \notin M\,\mathrm{Jac}(R)$ is minimal. So let $m_1, \ldots, m_l \notin M\,\mathrm{Jac}(R)$ and $m_j = n_j s_j$, $j > l$ for $n_j \in M$, $s_j \in \mathrm{Jac}(R)$. Let N' be the submodule of M generated by $m_1, \ldots, m_l, n_{l+1}, \ldots, n_k$. We prove that every finite tuple from N (in particular our original tuple) realizes in N' and M the same pp-type. This will be enough, since N' is a finitely generated submodule of a locally coherent module and hence N' is finitely presented.

Otherwise (since every pp-formula $\varphi(x_1, \ldots, x_k)$ over a uniserial ring is equivalent to a conjunction of pp-formulae $s \mid xt_1 + \ldots + xt_k$) $0 \neq m = m_1t_1 + \ldots + m_kt_k \in Ms \setminus N's$ for some $0 \neq s \in \mathrm{Jac}(R)$. Suppose that $t_i \notin \mathrm{Jac}(R)$ for some $i \leq l$. We may assume that $t_1 = 1 \notin \mathrm{Jac}(R)$. Then $m, m_2, \ldots, m_k$ is a new system of generators for N and $m \in M\,\mathrm{Jac}(R)$, a contradiction. Thus $t_i \in \mathrm{Jac}(R)$ for every $i \leq l$ and $m_j \in N'\,\mathrm{Jac}(R)$ for $j > l$, hence $m \in N't$ for some $0 \neq t \in \mathrm{Jac}(R)$. Since mR is not isomorphic to R_R, we have $\mathrm{ann}_R(m) = rR$ for some $0 \neq r \in \mathrm{Jac}(R)$. Thus $t \mid x \wedge xr = 0 \in pp_{N'}(m)$. Since $m \notin N's$, we have $sr = 0$ by Lemma 3.3. If $ns = m$ for $n \in M$ then clearly $nR \cong R_R$, a contradiction. $\square$

The following corollary shows in particular that every countably generated uniserial locally coherent module over an exceptional uniserial coherent ring is pure projective.

Corollary 5.7 *Let M be a countably generated locally coherent module over an exceptional uniserial coherent ring such that $M = M\operatorname{Jac}(R)$. Then M is pure projective.*

Proof. Clearly R_R is not a submodule of M, hence M is pure projective by Lemma 5.6. $\square$

Now we prove that there exists a unique uniserial countably generated pure projective module over an exceptional uniserial coherent ring.

Lemma 5.8 *Let M, N be countably generated locally coherent uniserial modules over an exceptional uniserial coherent ring. Then $M \cong N$ and M, N are pure projective. In particular if $\operatorname{Jac}(R)$ is countably generated, then $M \cong \operatorname{Jac}(R)$.*

Proof. M, N are pure projective by Lemma 5.6. We will construct an isomorphism from M to N using a back and forth construction. Indeed let $f : M_0 = m_0R \to N_0 = n_0R$ be an isomorphism of finitely generated submodules of M and N such that $f(m_0) = n_0$. We should extend f to arbitrary $m \in M$. Since M is locally coherent, we may assume that $ms = m_0$ for $0 \neq s \in \operatorname{Jac}(R)$ and also $sR = \{r \in R \mid mr \in M_0\}$. So $s \mid m_0$ and if $\operatorname{ann}_R(m_0) = rR$ then $sr \neq 0$ (otherwise $\operatorname{rann}_R(s) = rR$ and $mR \cong R$ is a pure submodule in M, a contradiction).

Also since $N = N\operatorname{Jac}(R)$, there is $0 \neq t \in \operatorname{Jac}(R)$ such that $n_0 \in Nt$. By Lemma 3.3 it follows that $s \mid n_0$ in N, hence $n_0 = ns$ for some $n \in N$. Then $f(m) = n$ gives the required extension.

Thus $M \cong N$. Also $\operatorname{Jac}(R)$ is locally coherent and uniserial. If it is countably generated, then $M \cong \operatorname{Jac}(R)$. $\square$

We will denote the unique uniserial countably generated pure projective R-module by V. For instance any countably generated not finitely generated right ideal of R is isomorphic to V.

Now we are ready to classify pure projective modules without projective direct summands.

Lemma 5.9 *Let M be a pure projective module over an exceptional uniserial coherent ring such that R_R is not a direct summand of M. Then $M \cong (R/rR)^{(\beta)} \oplus V^{(\gamma)}$.*

Proof. By Kaplansky's theorem we may assume that M is countably generated.

By Lemma 5.5, every $0 \neq m \in M$ realizes a pp-type generated by either $xr = 0$, $0 \neq r \in \mathrm{Jac}(R)$ or by $a \mid x \wedge xb = 0$, $a, b \in \mathrm{Jac}(R)$, $ab \neq 0$. Let us represent M as a union $M_0 \subset M_1 \subset \ldots$ of finitely generated (hence finitely presented) submodules.

We prove that every $0 \neq m \in M$ is contained in a direct summand isomorphic to either R/rR or V. If $pp_M(m)$ is generated by $xr = 0$, $0 \neq r \in \mathrm{Jac}(R)$, then $mR \cong R/rR$ is a direct summand. Otherwise $pp_M(m)$ is generated by $a \mid x \wedge xb = 0$ as above, in particular $m_0 = m = na$ for some $n \in M$. Let $N = nR + M_0$ be decomposed as $N = (R/rR)^n$. Then there are $n' \in N$, $0 \neq s \in R$ such that $n's' = m_0$ and $n' \notin N \,\mathrm{Jac}(R)$. Again if $pp_M(n')$ is generated by $xt = 0$, we are done. Otherwise write $n' = m_1 t$ for some $t \in \mathrm{Jac}(R)$, then $m_0 = m_1 ts'$ and clearly $m_1 \notin M_0$.

Following this either we find a direct summand R/rR in M containing m, or we construct a sequence $m = m_0, m_1, \ldots \in M$ such that 1) $m_{i+1} r_{i+1} = m_i$ for $0 \neq r_{i+1} \in \mathrm{Jac}(R)$ and 2) $m_{i+1} \notin M_i$.

Let V be generated by $m_0, m_1, \ldots$. We prove that V is pure in M. Otherwise for some $0 \neq n \in V$, $0 \neq s \in \mathrm{Jac}(R)$ we have $n \in Ms \setminus Vs$. Since V is uniserial and countably generated, we have $n \in Va$ for some $0 \neq a \in \mathrm{Jac}(R)$ and also $nb = 0$ for $0 \neq b \in \mathrm{Jac}(R)$. Then $a \mid x \wedge xb = 0 \in pp_V(n)$, hence $sb = 0$ by Lemma 3.3. Then for $n's = n$ we obtain $n'R \cong R_R$, a contradiction.

Let us show that M/V is locally coherent. Let $T' \subseteq M/V$ be finitely generated. Then T' is an image of a finitely generated submodule $T \subset M$, hence $T' \cong T/T \cap V$. If $V \subseteq T$, then $V \subseteq M_i$ for some i, a contradiction. Otherwise $T \cap V \subset V$, hence $T \cap V = T \cap m_i R$ for some i, and this is finitely generated (being the intersection of two finitely generated submodules of a locally coherent module). Thus T' is finitely presented.

If $mR \cong R \subseteq M/V$, it is a preimage of a copy R in M, a contradiction. Thus by Lemma 5.6 M/V is pure projective, hence $M \cong V \oplus M/V$. □

Corollary 5.10 *Let M be a countably generated locally coherent module over an exceptional uniserial coherent ring such that $M = \mathrm{Jac}(M)$. Then M is pure projective and $M \cong V^{(\gamma)}$ for the unique uniserial countably generated locally coherent module V.*

6 An example

First let us prove a technical claim.

Lemma 6.1 *Let M be a pure projective module over an exceptional uniserial coherent ring R. Let the pp-type $p = pp_M(m)$ for $0 \neq m \in M$ be generated by a pp-formula $ab' \mid xb'$ for some $a, b' \in \mathrm{Jac}(R)$, $ab' \neq 0$. Then for every $s \in R$ the pp-type of $n = ms$ in M is finitely generated. If $s \in \mathrm{Jac}(R)$ then $pp_M(n)$ is generated by $t \mid x$, $t \in R$.*

Proof. Let $q(y) = pp_M(n)$. If s is invertible in R then clearly $q(y)$ is generated by $ab' \mid ys^{-1}b'$. So we may assume that $0 \neq s \in \mathrm{Jac}(R)$.

Let $\mathrm{rann}_R(a) = bR$. Then (see Figure 2) $a \mid x + xb' = 0 \to xb = 0$, in particular $b \in b' \,\mathrm{Jac}(R)$. So if $s \in bR$ then $n = 0$ whose pp-type is generated by $x = 0$.

Let $s \notin b'R$. Suppose that $b' = st$ for some $t \in R$. Then $ab' \mid mb'$ i.e. $ast \mid nt$ implies that $ast \mid yt \in q(y)$ and also $s \mid y \in q(y)$. Thus $ast \mid yt \wedge s \mid y \in q(y)$ which is equivalent (see Figure 1) to $as \mid y$. Also we prove that $\mathrm{rann}_R(n) = \mathrm{rann}_R(as)$. Indeed clearly $\mathrm{rann}_R(as) \subseteq \mathrm{ann}_R(n)$. Suppose that $nh = 0$ for some $h \in R$, hence $msh = 0$. Then $ab' \mid xb' \to xsh = 0$, hence $ash = 0$ by Corollary 4.3. If $n' \in M$ be such that $n'as = n$, then $n'R \cong R_R$, hence n is included in a direct summand of M isomorphic to R_R and the pp-type of n is generated by $as \mid y$.

Now suppose that $s \in b'R \setminus bR$, say $s = b't$ for $t \in R$. If $m' \in M$ is such that $m'ab' = mb'$ then $m'ab't = mb't = ms = n$, hence $ab't \mid y \in q(y)$. Similarly it can be proved that $\mathrm{ann}_R(n) = \mathrm{rann}_R(ab't)$ hence $m'R \cong R_R$ is a direct summand. Indeed clearly $\mathrm{rann}_R(ab't) \subseteq \mathrm{ann}_R(n)$. If $nh = 0$ then $mb'th = nh = 0$, hence $ab'th = 0$ by Corollary 4.3 again, which is as desired. □

Now we give an example of a bad pure projective module over an exceptional uniserial coherent ring.

Proposition 6.2 *Let R be an exceptional uniserial coherent ring. Then there exists a pure projective module W over R such that W is not a direct sum of indecomposable modules.*

Proof. Let $0 \neq b_i \in \mathrm{Jac}(R)$ for $i \geq 0$ be such that $b_i \in b_{i+1} \,\mathrm{Jac}(R)$ and $\mathrm{lann}_R(b_i) = Ra_{i+1}$ for $0 \neq a_i \in \mathrm{Jac}(R)$. Clearly that $a_{i+1} \in \mathrm{Jac}(R)a_i$ and $\mathrm{rann}_R(a_{i+1}) = b_iR$. Hence the implication $a_ib_i \mid xb_i \to xb_{i-1} = 0$ is proper and the pair $(xb_{i-1} = 0/a_ib_i \mid xb_i)$ is minimal (see Figure 3).

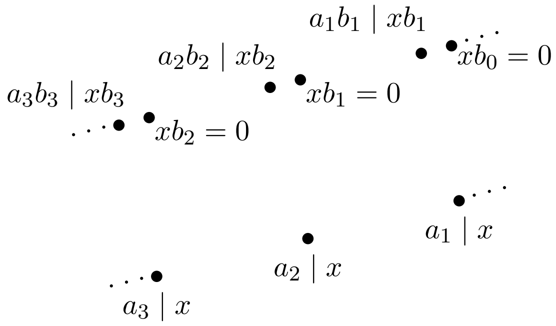

Figure 3:

Let $M_0 = R/b_0R$ and $m_0 = 1$. Then (M_0, m_0) is a free realization of the pp-formula $xb_0 = 0$. Let $M_1 = R \oplus R/b_1R$ and $m_1 = (0,1)$. We define the map $f_0 : M_0 \to M_1$ by $f_0(1) = (a_1, 1)$. Clearly $(M_1, f_0(m_0))$ is a free realization of $a_1b_1 \mid xb_1$. Since there are no other annihilator formulas between $a_1b_1 \mid xb_1$ and $xb_0 = 0$, this map is an embedding.

Similarly let $M_2 = R \oplus R \oplus R/b_2R$, $m_2 = (0,0,1)$ and let $f_1 : M_1 \to M_2$ be defined by $f_1(1,0) = (1,0,0)$ and $f_1(m_1) = f_1(0,1) = (0, a_2, 1)$. Then $(M_2, f_1(m_1))$ is a free realization of $a_2b_2 \mid xb_2$ in M_2.

Also $f_1f_0(m_0) = (a_1, a_2, 1)$ has pp-type generated by $a_1 \mid x + a_2 \mid x + xb_2 = 0$ which is clearly equal to $a_1b_1 \mid xb_1$.

Similarly let $M_i = R^i \oplus R/b_iR$, $m_i = (\bar{0}, 1)$ and let $f_i : M_i \to M_{i+1}$ be defined by $f(\bar{r}, 0) = (\bar{r}, 0, 0)$ for $\bar{r} \in R^i$ and $f(m_i) = f(\bar{0}, 1) = (\bar{0}, a_{i+1}, 1)$. Clearly (see Figure 3) every f_i is an embedding.

Let $W = \varinjlim(M_i, f_i)$. As above it is not difficult to see that $m_i = (\bar{0}, 1) \in M_i$ realizes in W the pp-type generated by $a_{i+1}b_{i+1} \mid xb_{i+1}$. Indeed the image of m_i in M_{i+1} is $(\bar{0}, a_{i+1}, 1)$ which is a free realization of $a_{i+1}b_{i+1} \mid xb_{i+1}$. Also the further image in M_{i+2} is $(\bar{0}, a_{i+1}, a_{i+2}, 1)$ which is a free realization of $a_{i+1} \mid x + a_{i+2} \mid x + xb_{i+2} = 0$ which is equal to $a_{i+1}b_{i+1} \mid xb_{i+1}$ and so on.

We prove that there is no element m in W whose pp-type is generated by $xb = 0$ for some $0 \neq b \in \mathrm{Jac}(R)$ or $a \mid x \wedge xb = 0$ for some $a, b \in \mathrm{Jac}(R)$, with $ab \neq 0$. Indeed let $m = (\bar{r}, s) \in M_i = R^i \oplus R/b_iR$.

Then $m = m_1 + m_2$ where $m_1 = (\bar{r}, 0)$ and $m_2 = (\bar{0}, s)$. Clearly $pp_W(m_1)$ is generated by a pp-formula $t \mid x$ for some $t \in R$ and if φ generates $pp_W(m_2)$, then $p = pp_M(m)$ is generated by $t \mid x + \varphi$ so we can not reach a forbidden formula if φ is not such.

So it suffices to prove that $pp_W(0, s)$ is not of this form. For $s = 1$ the pp-type of m in M is generated by $a_{i+1}b_{i+1} \mid xb_{i+1}$ which is as desired and for invertible s we can use Lemma 6.1. If $s \in b_iR$, then $s = 0$ in R/b_iR. Suppose that $s \in \mathrm{Jac}(R) \setminus b_iR$, then by Lemma 6.1 the pp-type of s in M_{i+1} is generated by a pp-formula $v \mid x$, so the same is true in W.

We claim that W is pure projective. By Fact 2.3 it suffices to prove that W is pp-atomic (i.e. realizes only finitely generated pp-types). In fact we show that for any $\overline{m} = (m_1, \ldots, m_k) \in M_i$ we have $pp_W(\overline{m}) = pp_{M_{i+1}}(\overline{m})$. Since M_{i+1} is finitely presented that will give the desired. So let $m = m_1t_1 + \ldots + m_nt_n \in M_i$ and $t \mid x$ in W but $\neg\, t \mid x$ in M_{i+1} for some $0 \neq t \in \mathrm{Jac}(R)$. Let us write $m = (\bar{r}, s)$ for $s \in R/b_iR$. In particular, $t \mid m$ in W implies $t \mid \bar{r}$ in R^i, hence we can remove this component. If $s \in \mathrm{Jac}(R)$ then by Lemma 6.1 the type of s in M_i is maximal, hence $t \mid s$ in M_{i+1}, a contradiction.

The remaining case is that s in invertible, and so we may assume that $s = 1$. Then the pp-type of 1 in W and M_{i+1} is generated by $a_{i+1}b_{i+1} \mid xb_{i+1}$ which also yields the desired result. Thus W is pure projective.

Suppose that $W = \oplus_{i \in I} M_i$ where M_i are indecomposable pure projective modules. By what has just been proved every pp-type realized in M_i is generated by either $s \mid x$ or by $ab \mid xb$ for $a, b \in \mathrm{Jac}(R)$, $ab \neq 0$. In both cases by Lemma 5.5, M_i contains a direct summand isomorphic to R_R, hence $M_i \cong R_R$, and so $M \cong R^{(\alpha)}$. But then every $m \in M$ has pp-type generated by $s \mid x$ for some $s \in R$ and we could not obtain, for example $a_1b_1 \mid xb_1$, a contradiction.

□

Facchini [5, Probl. 10] asked whether every direct summand of a serial module is serial. Also let us cite the first question in [5, Probl. 11]: Is every pure projective module over a serial ring serial? The following is a negative answer to both these questions.

Theorem 6.3 *Over an arbitrary exceptional uniserial coherent ring R there exists a pure projective module W that is not a direct sum of indecomposable modules.*

Proof. This follows from Proposition 6.2. □

We derive from this the following corollary.

Corollary 6.4 *There exists a ring with exactly three maximal right (left, twosided) ideals with a projective (countably generated) right module P such that P is not a direct sum of indecomposable modules.*

Proof. Let R, W be as in Theorem 6.3, $N = R \oplus R/rR$ and $S = \mathrm{End}(N)$. Then there is a correspondence between right pure projective modules over R and projective right S-modules. The projective S-module P that corresponds to W is as desired. □

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