SUPERDECOMPOSABLE PURE-INJECTIVE MODULES EXIST OVER SOME STRING ALGEBRAS

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Abstract. We prove that over every non-domestic string algebra over a countable field there exists a superdecomposable pure-injective module.

1. Introduction

There is a well-known dichotomy for the behavior of a finite dimensional algebra A over a field $\Bbbk$. Roughly speaking A is tame if a description of all finite dimensional A-modules is available, otherwise A is wild. This definition can be made precise, and then Drozd's theorem states (at least for an algebraically closed $\Bbbk$) that every finite dimensional algebra is either tame or wild but not both.

Unfortunately, the usual definition of tameness and wildness refers to some infinite dimensional A-modules (what Ringel [10] describes as an 'external structure') so it would be nice to find one appealing only to finite dimensional representations. It has been conjectured by Prest [6, Ch. 13] (see also [10, p. 38] and a discussion in [5, p. 219]) that A is tame if and only if A does not posses a superdecomposable (i.e. without indecomposable direct summands) pure-injective module. This means just that every direct product of indecomposable finite dimensional A-modules contains an indecomposable direct summand.

In this paper we refute this conjecture by proving that over an arbitrary non-domestic string algebra over a countable field there exists a superdecomposable pure-injective module. This class of algebras is well known to be tame and includes among others the Gelfand–Ponomarev algebras as well

2000 *Mathematics Subject Classification.* 16G20, 16D50.

Key words and phrases. Pure-injective module, string algebra, superdecomposable module.

This paper was written during the visit of the author to the University of Manchester supported by EPSRC grant GR/R44942/01. He would like to thank the University for the kind hospitality.

as the dihedral algebras. So it seems now that the classification of pure-injective modules over a non-domestic string algebra (just slightly touched on by Baratella and Prest [1]) is a more challenging problem than previously believed.

Is countability of $\Bbbk$ necessary in the above result? In fact, our main result does not appeal to any countability assumption: we prove that the lattice of all pp-formulae over any non-domestic string algebra does not have width. But to extract a superdecomposable pure-injective module from this we need an ingenious construction of Ziegler [12] that seems to work only if $\Bbbk$ is countable.

Note that the existence of a superdecomposable pure-injective module over a Gelfand–Ponomarev algebra was posed as a problem in Jensen and Lenzing [5] (see Remark 8.72 and Problem 13.28). So we give a partial, i.e. over a countable field, answer to this question. The reader may also consult [5] to see how to construct a superdecomposable pure-injective module over many (conjecturally all) wild finite dimensional algebras.

All the machinery used in the proofs is quite well known. Prest [7] was the first to notice that over the dihedral (and many similar) algebras there exists a densely ordered chain of morphisms between string modules. In other words the lattice of all pp-formulae over these algebras does not have m-dimension. This result (with a similar proof) was extended by Schröer [11] to an arbitrary non-domestic string algebra.

It is also well known (see Ringel [8] for a detailed explanation) that over a dihedral algebra there are two natural chains of proper morphisms between indecomposable finite dimensional modules. All we have noticed is that the (distributive) lattice generated by these two chains is generated freely, therefore its width is undefined.

I am indebted to Mike Prest for his helpful suggestions and interest.

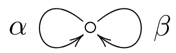
2. Preliminaries

Quite a few model theoretic terms, which appear in what follows, can be found in [6]. Otherwise, as is explained in [7], one could always replace the term ‘pp-formula’ by ‘pointed finitely presented module’, and the term ‘implication between pp-formulae’ by ‘morphism between pointed modules’. All the modules in the sequel will be left modules.

Let A be a finite dimensional algebra given by a quiver with monomial relations. For an arrow α, we will denote by $s(\alpha)$ the starting point of α and by $e(\alpha)$ the ending point of α. Also, for every arrow α, we consider its formal inverse α^{-1} as an arrow going into an opposite direction. Thus $e(\alpha^{-1}) = s(\alpha)$ and $s(\alpha^{-1}) = e(\alpha)$.

A is said to be a *string algebra* if the following holds true: 1) every vertex is a starting point for at most two arrows and the ending point for at most two arrows; 2) given an arrow α, there is at most one arrow β such that $e(\beta) = s(\alpha)$, and the composition $\alpha\beta$ is not a relation in A (i.e. nonzero in A); 3) given an arrow α, there is at most one arrow γ such that $e(\alpha) = s(\gamma)$, and the composition $\gamma\alpha$ is not a relation in A.

For instance, the Gelfand–Ponomarev algebra $G_{n,m}$ is the path algebra of the quiver

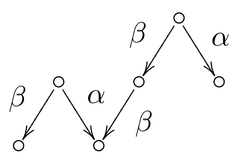


with relations $\alpha^n = \beta^m = 0$. It is well known that $G_{n,m}$ is tame non-domestic if $m + n \geq 5$.

Let A be a string algebra. A string C over A is a sequence $c_1 \dots c_n$ with the following properties: 1) for every i, either $c_i = \alpha$ is a (direct) arrow, or $c_i = \alpha^{-1}$ is an inverse arrow; 2) $s(c_i) = e(c_{i+1})$ for every $1 \leq i \leq n-1$; 3) $c_i \neq c_{i+1}^{-1}$ for every $1 \leq i \leq n-1$; 4) neither $c_i \dots c_{i+t}$ (direct arrows) nor $c_{i+t}^{-1} \dots c_i^{-1}$ (inverse arrows) is a relation in A for $1 \leq i \leq i+t \leq n$.

Given a string $C = c_1 \dots c_n$, we define a string module $M(C)$ in the following way. The $\Bbbk$-basis for $M(C)$ is given by vectors $z_0, \dots, z_n$. If $c_i = \alpha$ is direct, then set $\alpha z_i = z_{i-1}$, and if $c_i = \beta^{-1}$ is inverse, then put $\beta z_{i-1} = z_i$. All the remaining actions are defined to be zero. Following [11] we draw direct arrows from upper right to lower left and inverse arrows from upper left to lower down.

For instance,



is a string module over $G_{2,3}$ corresponding to the string $\beta\alpha^{-1}\beta^2\alpha^{-1}$.

By [2] all string modules over a string algebra are indecomposable.

Let $C = c_1 \dots c_n$, $n \geq 1$ be a string. What are the possible ways to extend this string to a string $c_1 \dots c_n c_{n+1}$? Suppose that c_n is a direct letter α. If c_{n+1} is a direct letter β, then β ends in the vertex where α starts. Since $\alpha\beta$ is a string, there is only one possibility for β (such that $\alpha\beta$ is not a relation in A). On the other hand, if c_{n+1} is an inverse letter γ^{-1}, then α and γ start at the same vertex (and $\alpha \neq \gamma$ since Cc_{n+1} is a string). Since there are at most two arrows starting in the given vertex, γ is uniquely defined. Moreover, if both β and γ are defined, then $\alpha\beta \neq 0$ implies $\gamma\beta = 0$.

Now we define a (linear) order $<$ on the set of strings with the same first (direct) letter. For strings B and C we put $B < C$ if one of the following holds true 1) $B\gamma^{-1}D = C$ for some γ, D; 2) $B = C\beta E$ for some β, E; or 3) $B = B'\beta F$, $C = B'\gamma^{-1}G$ for some B', F, γ and G. Thus, to compare two strings we look at their common initial part (by assumption, there is at least one letter in this part), and compare letters following this part.

Note that, if $B < C$ by 1), then $B < CS$ for arbitrary S (such that CS is a string). Similarly, if $B < C$ by 2), then $BT < C$ for any T. Finally, if $B < C$ by 3), then $BU < CV$ for all U and V.

Lemma 2.1. *Let $B < C$ be strings with first letter α, and let $M(B)$, $M(C)$ be the corresponding string modules. Then there exists a (canonical) morphism $f : M(C) \to M(B)$ such that $f(z_0) = z_0$. Moreover, every such morphism is proper, meaning that there is no morphism $g : M(B) \to M(C)$ such that $g(z_0) = z_0$.*

Proof. This is just a graph map in the sense of [3]. Let us include the description for completeness.

Note that for every string $B = b_1 \dots b_n$ there is a canonical embedding $M(B) \to M(B\beta D)$ given by $z_i \to z_i$, $i \leq n$, and also a canonical projection $M(B\gamma^{-1}E) \to M(B)$ given by $z_i \to z_i$, $i \leq n$ and $z_j \to 0$ for $j > n$.

Now, if $B < C$ by 1), then define $f : M(C) \to M(B)$ to be the projection $M(C) = M(B\gamma^{-1}D) \to M(B)$. If $B < C$ by 2), then define $f : M(C) \to M(B)$ to be the embedding $M(C) \to M(C\beta E) = M(B)$. For 3) we obtain f as a composite map $M(C) = M(B'\gamma^{-1}G) \to M(B') \to M(B'\beta F) = M(B)$.

Suppose that there exists a map $g : M(B) \to M(C)$ such that $g(z_0) = z_0$. Note that $M(C)$ is indecomposable and pure-injective (being of finite dimension). By [6, Prop. 4.26], every noninvertible endomorphism of $M(C)$

strongly increases the pp-type of every nonzero element. Since $gf(z_0) = z_0$, it follows that gf is invertible. Therefore $M(B) = \mathrm{im}(f) \oplus \ker(g)$. Since $M(B)$ is indecomposable, and $f, g \neq 0$, we conclude that f is epi and g is mono. By symmetry, f is mono and g is epi, hence $M(C) \cong M(B)$, a contradiction. □

Since $M(C)$ is finitely presented, by [6, Prop. 8.4] there is a (positive primitive) formula $\varphi_C(x)$ which generates the pp-type of z_0 in $M(C)$.

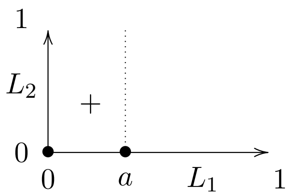
Corollary 2.2. *Let $B < C$ be strings with first letter α. Then $\varphi_B \to \varphi_C$ and this implication is proper.*

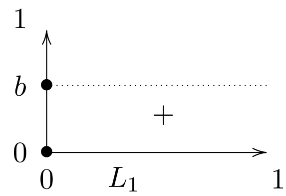
Proof. Let p be the pp-type of z_0 in $M(B)$, and let q be the pp-type of z_0 in $M(C)$. Then p is generated by φ_B and q is generated by φ_C. By Lemma 2.1, there is a morphism $f : M(C) \to M(B)$ such that $f(z_0) = z_0$. It follows that $q \subseteq p$, hence $\varphi_B \to \varphi_C$. Suppose that $\varphi_C \to \varphi_B$. Then there exists a morphism $g : M(B) \to M(C)$ such that $g(z_0) = z_0$, which contradicts Lemma 2.1. □

Clearly φ_C can be chosen of the form $\exists\, z_1, \dots, z_n$, followed by a complete description of the action on z_i. In fact, for $1 \leq i \leq n-1$ it suffices to describe only the action given by c_i and c_{i+1}. Indeed, suppose we have $\alpha^{-1} z_i \beta$ in $M(C)$, i.e. α and β are different arrows ending in z_i. Then, for every arrow γ starting in z_i, we have either $\gamma\alpha = 0$ or $\gamma\beta = 0$, hence γz_i must be zero. If $c_1 = \alpha$ is a direct arrow, and there exists γ such that $\gamma\alpha \neq 0$, then the formula should, in addition, say $\gamma z_0 = 0$. Similarly, we may need to add one more relation for z_n.

3. Width

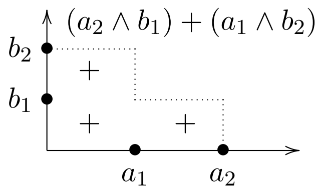
Let L_1 and L_2 be chains with 0 and 1. By $L = L_1 \otimes L_2$ we will denote the modular lattice freely generated by L_1 and L_2. It is well known (see [4, Th. 13]) that this lattice is distributive. Moreover, it is quite easy to visualize the structure of L. Let $L_1 \times L_2$ be presented as a plane. We assign to $a \in L_1$ the vertical strip $[0; a] \times L_2$, and to $b \in L_2$ the horizontal strip $L_1 \times [0; b]$:





It follows from [4, proof of Th. 13] that the lattice generated by these strips (with respect to ordinary meet and joint operations) is isomorphic to L. To be more precise, we should avoid $1 \in L_1$ and $1 \in L_2$ being glued together via this representation. This can be overcome as follows: add formally ∞ after 1 in L_1, and the same for L_2.

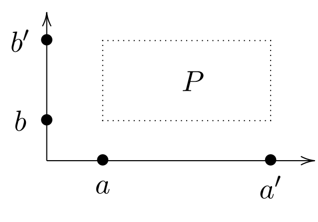
Note that an element $a \wedge b$ is represented as a rectangle $[0; a] \times [0; b]$. Since L is distributive, and L_1, L_2 are chains, every element $l \in L$ can be written as $l = (a_n \wedge b_1) + (a_{n-1} \wedge b_2) + \cdots + (a_1 \wedge b_n)$, where $a_1 < \cdots < a_n \in L_1$ and $b_1 < \cdots < b_n \in L_2$ (we use $+$ instead of $\vee$). Thus a typical element of L looks like a descending ladder (where the first step may be of infinite height and the last of infinite length):



Note that we can rearrange brackets in l: $l = a_n \wedge (b_1 + a_{n-1}) \wedge \cdots \wedge (b_{n-1} + a_1) \wedge b_n$.

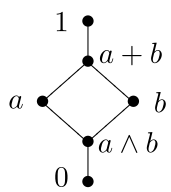
Given $a, b \in L$, (a/b) will denote the interval $[a \wedge b; a]$.

Let $a < a' \in L_1$, $b < b' \in L_2$. Then the interval $(a' \wedge b'/a + b) = [a' \wedge b' \wedge (a + b); a' \wedge b'] = [(a \wedge b') + (a' \wedge b); a' \wedge b']$ can be thought as the set theoretical difference of the representing figures, i.e. as a rectangle $P = [a; a'] \times [b; b']$ (caution: P is not an element of L):



Given a lattice L with 0 and 1, we define the 2-dimension of L, $2\dim(L)$, by induction on the ordinals. Put $2\dim(L) = 0$ iff $0 = 1$ in L, i.e. L consists of one element. Set $2\dim(L) = \alpha$, if $2\dim(L)$ is not less than α, and for every $a \in L$ we have either $2\dim(a/0) < \alpha$ or $2\dim(1/a) < \alpha$. For instance, $2\dim(2^n + 1) = n + 1$ and $2\dim(\omega + 1) = \omega$. Thus $2\dim(L)$ shows how many times we can split L into two 'equal' pieces. Note that $2\dim(L)$ is defined iff L does not contain the rationals as a suborder.

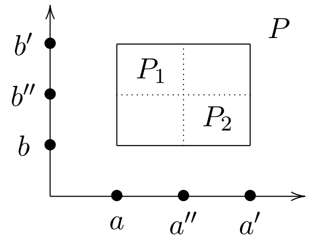
Let L be a (modular) lattice with 0 and 1. The width of L, $\mathrm{w}(L)$, will be defined by induction on the ordinals. Set $\mathrm{w}(L) = 0$ iff $0 = 1$ in L. Now put $\mathrm{w}(L) = \alpha$, if $\mathrm{w}(L)$ is not less then α, and for every $a, b \in L$ we have either $(a + b/a) < \alpha$ or $(a + b/b) < \alpha$. For instance, if L is a chain with at least two elements, then $\mathrm{w}(L) = 1$. The width of L estimates how many 'diamonds' can be repeatedly embedded in L. In particular, $\mathrm{w}(L) \geq \alpha + 1$ if there are $a, b \in L$ such that 'α diamonds' can be nested in each of the intervals $(a + b/a)$ and $(a + b/b)$, hence also in $(a/a \wedge b)$ and $(b/a \wedge b)$:



Proposition 3.1. *Let L_1 and L_2 be chains with 0 and 1 such that $2\dim(L_1) = \alpha$, $\dim(L_2) = \beta$, and let $L = L_1 \otimes L_2$. Then $\mathrm{w}(L) \geq \min(\alpha, \beta)$.*

Proof. Let $a < a' \in L_1$, $b < b' \in L_2$ be such that $2\dim(a'/a) \geq \alpha$, $2\dim(b'/b) \geq \beta$ and $\alpha, \beta \geq \gamma$. By induction on γ we prove that the width of the interval $(a' \wedge b'/a + b)$ (it looks like the rectangle $P = [a; a'] \times [b; b']$ in the figure below) is not less than γ.

The proof is obvious when γ is a limit. So let $\gamma = \delta + 1$. Since $2\dim(a'/a) \geq \gamma$, there exists an $a'' \in (a'/a)$ such that $2\dim(a''/a) \geq \delta$ and $2\dim(a'/a'') \geq \delta$. Similarly there exists an $b'' \in (b'/b)$ such that $2\dim(b''/b) \geq \delta$ and $2\dim(b'/b'') \geq \delta$:



We prove that there are $\theta_1, \theta_2 \in L$ such that

1) $(a' \wedge b') \wedge (a+b) < \theta_1, \theta_2 < a' \wedge b'$;

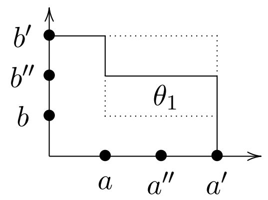
2) the interval $(\theta_1 + \theta_2/\theta_1)$ is isomorphic to the interval $(a'' \wedge b'/a + b'') = (a'' \wedge b')/(a'' \wedge b'') + (a \wedge b')$ (it looks like the rectangle $P_1 = (a''/a) \times (b'/b'')$ on the figure);

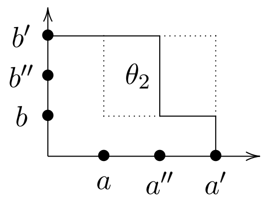
3) the interval $(\theta_1 + \theta_2/\theta_2)$ is isomorphic to the interval $(a' \wedge b''/a'' + b) = (a' \wedge b'')/(a' \wedge b) + (a'' \wedge b'')$ (it looks like the rectangle $P_2 = (a'/a'') \times (b''/b)$ on the figure).

Then, by the induction hypothesis, we would have $\mathrm{w}(P_1) \geq \delta$, $\mathrm{w}(P_2) \geq \delta$, therefore $\mathrm{w}(P) \geq \gamma$.

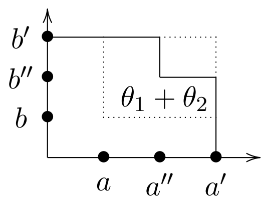
It is quite easy to prove this on the level of figures (just insert P_1 and P_2 in P) but the lattice theoretical proof is comparatively harder.

Take $\theta_1 = (a' \wedge b'') + (a \wedge b') = a' \wedge (b'' + a) \wedge b'$ and $\theta_2 = (a' \wedge b) + (a'' \wedge b') = a' \wedge (b + a'') \wedge b'$:

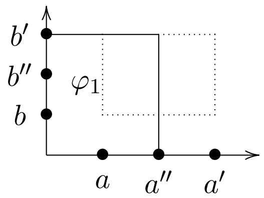


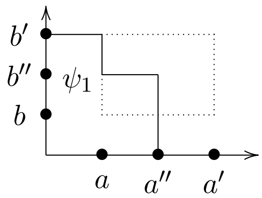


In particular, θ_1 and θ_2 are incomparable, and $a' \wedge b' \wedge (a+b) < \theta_1, \theta_2 < a' \wedge b'$. Clearly we have $\theta_1 + \theta_2 = (a' \wedge b'') + (a'' \wedge b') = a' \wedge (b'' + a'') \wedge b'$:



To prove 2), take $\varphi_1 = a'' \wedge b'$ and $\psi_1 = (a'' \wedge b'') + (a \wedge b')$:

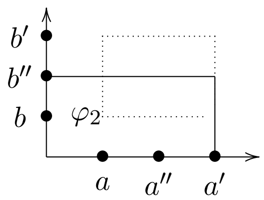


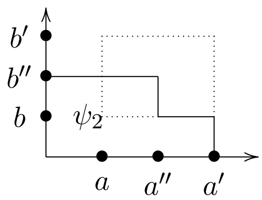


Then $\psi_1 < \varphi_1$, and (φ_1/ψ_1) is a rectangle P_1. To prove that $(\varphi_1/\psi_1) \cong (\theta_1 + \theta_2/\theta_1)$ it suffices to check that $\varphi_1 + \theta_1 = \theta_1 + \theta_2$ and $\varphi_1 \wedge \theta_1 = \psi_1$.

But this is easily seen from the above figures.

To prove 3), let $\varphi_2 = a' \wedge b''$ and $\psi_2 = (a' \wedge b) + (a'' \wedge b'')$:





Then $\psi_2 < \varphi_2$, and (φ_2/ψ_2) is a rectangle P_2.

To prove that $(\varphi_2/\psi_2) \cong (\theta_1 + \theta_2/\theta_2)$, it suffices to check that $\varphi_2 + \theta_2 = \theta_1 + \theta_2$ and $\varphi_2 \wedge \theta_2 = \psi_2$. This is also evident from the above figures. □

Corollary 3.2. *Let L_1 and L_2 be lattices with 0 and 1 such that both $2\dim(L_1)$ and $2\dim(L_2)$ are undefined. Then $\mathrm{w}(L_1 \otimes L_2)$ is undefined.*

4. The main result

We are in a position to prove the main result of the paper.

Theorem 4.1. *Let A be a non-domestic string algebra. Then the width of the lattice of all pp-formulae over A is equal to infinity.*

Proof. Recall that a *band* is a string $C = c_1 \dots c_n$ of length ≥ 2 such that 1) all powers C^m are defined; 2) C is not a power of a string of smaller length; 3) c_1 is a direct letter and c_n is an inverse letter. If $c_1 = \alpha$ and $c_n = \beta^{-1}$, then 1) yields that α and β are different arrows ending in the same vertex. Thus β is uniquely determined by α.

Following [11], for an arrow α, by $B(\alpha)$ we denote the set of all bands with first letter α. By what we have said above, the last letter β^{-1} is the same for all bands in $B(\alpha)$. Since A is not domestic, by [11, p. 41] there is an α for which there are two different bands $B, C \in B(\alpha)$, such that B and C contain no substring of the form $\beta^{-1}\alpha$.

Let $B = b_1 \dots b_n$ and $C = c_1 \dots c_m$, where we may assume that $B < C$. Note that the case 2) (see the definition of $<$ above) is not possible. Indeed, then $B = C\delta E$, hence $\delta = \alpha$, and B contains a substring $\beta^{-1}\alpha$, a contradiction. Thus either 1) $C = B\gamma^{-1}D$ or 3) $B = B'\delta F$ and $C = B'\gamma^{-1}G$. In both cases $BC < CT$ for all T. Indeed, this is clear for 3) (see remark above). For 1) it is also clear, since $BC = B\alpha C'$ and $CT = B\gamma^{-1}DT$.

Thus (see also [7, p. 450]) we obtain the following strongly descending chain of strings (every word consisting of the letters B and C is a string):

$$BC > BCBC > BCB^2C > BCB^2CBC > BCB^2CB^2C > B^2C\,.$$

The interval $[B^2C; BC]$ is isomorphic to $[BCB^2C; BCBC]$ (after removing BC from the beginning) and $[BCB^2CB^2C; BCB^2CBC]$ (after removing BCB^2C from the beginning). Thus we may extend this construction to obtain a dense subchain L_1 of $<$ with 0 and 1.

Recall that for each $T \in L_1$ we have defined a pp-formula $\varphi_T(x)$. By Lemma 2.2, for all $T < U \in L_1$ we have $\varphi_T \to \varphi_U$, and this implication is proper. Thus we obtain a dense chain of pp-formulae over A (we will use L_1 also to denote this chain).

Now let us consider the words B^{-1} and C^{-1}. The first letter of both words is β, the last letter is α^{-1} and both B and C contain no substring $\alpha^{-1}\beta$. Thus we may repeat our constructions to obtain a densely ordered chain L_2 (with 0 and 1) of words constructed from the letters B^{-1} and C^{-1}. Note that every word in L_2 has β as a starting arrow and $\beta \neq \alpha$. As above we may consider L_2 as a densely ordered chain of pp-formulae ψ_S such that $\psi_S \to \psi_T$ iff $S < T$.

For every $V \in L_2$, $U \in L_1$ we may consider a string module $M = M(V^{-1}U)$ defined by the word $V^{-1}U$ with the basis $z_{-m}, \dots, z_0, z_1, \dots, z_n$, such that z_0 is located between V^{-1} and U (in particular $\beta z_{-1} = z_0$ and $\alpha z_1 = z_0$). It is easy to check that (M, z_0) is a free realization of $\varphi_U \wedge \psi_V$ (it is just the amalgam of modules $M(U)$ and $M(V)$ by the submodule generated by z_0; the other way to see this is to construct for every element n of a module N with $N \models (\varphi_U \wedge \psi_V)(n)$ a morphism $f : M \to N$ such that $f(z_0) = n$).

Let L be generated by L_1 and L_2 in the lattice of all pp-formulae over A. We prove that L does not have width (then the lattice of all pp-formulae over A does not have width). By Lemma 3.1, it suffices to check that L is freely generated by L_1 and L_2. Considering the canonical forms of elements in L, it remains to prove the following: if $T < U \in L_1$ and $S < V \in L_2$, then $\varphi_U \wedge \psi_V$ does not imply $\varphi_T + \psi_S$.

Assume to the contrary that $\varphi_U \wedge \psi_V$ implies $\varphi_T + \psi_S$. Let $M = M(V^{-1}U), z_0$ be chosen (as above) to be a free realization of $\varphi_U \wedge \psi_V$. It is clear (see a similar proof in [8, p. 26]) that $\varphi_T(M)$ and $\psi_S(M)$ are homogeneous spaces, i.e. $\sum_i \lambda_i z_i \in \varphi_T(M)$ iff $z_i \in \varphi_T(M)$ for every i with $\lambda_i \neq 0$. Thus $z_0 \in \varphi_T(M)$ or $z_0 \in \psi_S(M)$ yields that either $\varphi_U \wedge \psi_V \to \varphi_T$ or $\varphi_U \wedge \psi_V \to \psi_S$.

If $\varphi_U \wedge \psi_V \to \psi_T$, then there is a morphism $f : M \to M(T)$ such that $f(z_0) = z_0$. Restricting f to the submodule $M(U)$, we obtain a contradiction to Lemma 2.1. Similarly $\varphi_U \wedge \psi_V \to \psi_S$ yields (since $M(V)$ is a submodule of M) $\psi_V \to \psi_S$, a contradiction again. □

Corollary 4.2. *Let A be a (finite dimensional) non-domestic string algebra over a countable field. Then there exists a superdecomposable pure-injective A-module.*

Proof. By Theorem 4.1 the width of the lattice of all pp-formulae over A is undefined. It remains to apply [6, Thm. 10.13]. □

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