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ORDINARY DIFFERENTIAL EQUATIONS

Existence of Weak Solutions of Stochastic Differential Equations with Discontinuous Coefficients and with a Partially Degenerate Diffusion Operator

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We study the existence of weak solutions of stochastic differential equations

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW(t), \qquad X \in R^d, \tag{1}$$

with Borel measurable functions $f : R_+ \times R^d \to R^d$ and $g : R_+ \times R^d \to R^{d\times d}$, where $W(t)$ is a d-dimensional Brownian motion.

The aim of the present paper is to weaken known conditions on the functions f and g providing the existence of weak solutions of Eq. (1).

The first existence theorem for weak solutions was obtained in [1] under the assumption that f and g are continuous bounded functions. It was shown in [2] that, for weak solutions to exist, it is sufficient that f and g are measurable bounded functions and g is a nondegenerate matrix ($\lambda^{\mathrm T} g g^{\mathrm T} \lambda \ge \nu\|\lambda\|^2$, $\nu > 0$, for all $\lambda \in R^d$). Then the nondegeneracy condition for the matrix g was weakened. It was shown in [3] that the system

$$\begin{aligned} dx(t) &= f^{(1)}(t, x(t), y(t))dt + g^{(1)}(t, x(t), y(t))dW(t), \\ dy(t) &= f^{(2)}(t, x(t), y(t))dt + g^{(2)}(t, x(t), y(t))dW(t), \qquad x \in R^l, \qquad y \in R^{d-l}, \end{aligned} \tag{2}$$

has weak solutions under the following assumptions: the functions $f^{(1)}$, $f^{(2)}$, $g^{(1)}$, and $g^{(2)}$ are Borel measurable and bounded and continuous with respect to y, and $g^{(1)}$ is a nondegenerate matrix. A similar theorem was proved in [4]. It was shown in [5] that Eq. (1) has weak solutions if f and g are measurable functions and have a linear growth as $\|X\| \to \infty$, and the closure of the intersection of the weak degeneracy set of the mapping g, that is, the set $\Big\{(t, X) \mid \int_{U(t,X)} (\det g g^{\mathrm T}(\tau, y))^{-1} d\tau\, dy = \infty$ for each open neighborhood $U(t, X)$ of a point $(t, X)\Big\}$, with the set of points of discontinuity of the function f or g is contained in the set of zeros of the mappings f and g.

In the present paper, we prove an existence theorem for weak solutions of Eq. (1), which, in the case of system (2), can be stated as follows: if the functions $f^{(1)}$, $f^{(2)}$, $g^{(1)}$, and $g^{(2)}$ are Borel measurable and locally bounded and continuous with respect to y and the set $H \times R^{d-l}$ is contained in the set of points of continuity of the functions f and $gg^{\mathrm T}$, where $H = \Big\{(t, x) \in R_+ \times R^l \mid$ for each open neighborhood $U(t, x)$ of the point (t, x), there exists an $a > 0$ such that the integral $\int_{U(t,x)} \sup_{y\in R^{d-l},\ \|y\|\le a} \big(\det g^{(1)} g^{(1)\mathrm T}(t, x, y)\big)^{-1} dt\, dx$ is either undefined or equal to $\infty\Big\}$, then system (2) has a weak solution.

We use the following notation: $a \wedge b$ is the minimum of numbers a and b; $a \vee b$ is the maximum of numbers a and b; P^x is the probability distribution of a random variable x; the relation $P^x = P^y$ means that the distributions of random variables x and y coincide; $E(x)$ is the expectation of a random variable x; f^i is the ith component of a vector function f; g^{ij} is the (i,j)th entry of a matrix function g; 1_A is the characteristic function of a set A;

$$\|X\| = \|(x_1, \dots, x_d)\| = \left(x_1^2 + \dots + x_d^2\right)^{1/2};$$

a.s. stands for "almost surely"; $C, C_1, C_2, \dots$ are universal constants; $B(0,r) = \left\{x \in R^d \mid \|x\| \le r\right\}$.

Definition. Suppose that there exists a process $X(t)$ defined on some probability space $(\Omega, \mathscr{F}, P)$ with a flow $\mathscr{F}_t$ of σ-algebras such that the following assertions are valid.

1. There exists an $(\mathscr{F}_t)$-stopping time e such that the process $X(t)1_{[0,e)}(t)$ is $(\mathscr{F}_t)$-coordinated and has continuous trajectories for $t < e$ a.s. and $\limsup_{t\uparrow e} \|X(t)\| = \infty$ if $e < \infty$.

2. There exists an $(\mathscr{F}_t)$-Brownian motion $W(t)$ with $W(0) = 0$ a.s.

3. The processes $f(t, X(t))$ and $g(t, X(t))$ belong to the spaces L_1^{loc} and L_2^{loc}, respectively, where L_i^{loc} is the set of all measurable $(\mathscr{F}_t)$-coordinated processes ψ such that $\int_0^t \|\psi(s,\omega)\|^i ds < \infty$ a.s. for each $t \ge 0$, $i \in \{1, 2\}$.

4. The relation

$$X(t) = X(0) + \int\limits_0^t f(\tau, X(\tau))d\tau + \int\limits_0^t g(\tau, X(\tau))dW(\tau)$$

is valid with probability 1 for all $t \in [0, e)$. Then the tuple $(\Omega, \mathscr{F}, P, \mathscr{F}_t, W(t), X(t), e)$ [or, briefly, $X(t)$] is called a *weak solution* of Eq. (1).

The matrix $\sigma(t,X) = g(t,X)g^{\mathrm{T}}(t,X)$ is symmetric and nonnegative. There exist Borel measurable orthogonal diagonal matrices T and $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_d)$, respectively, such that $\sigma = T\Lambda T^{\mathrm{T}}$. Let $g^* = T \operatorname{diag}\left(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_d}\right)$. Without loss of generality, we assume that $g = g^*$ in system (1) [6, pp. 97–98 of the Russian translation].

We take the rows of the matrix g with indices $\beta_1, \dots, \beta_l$. Then

$$\sigma_{\beta_1,\dots,\beta_l}(t, x_1, \dots, x_d) = \operatorname{col}(g_{\beta_1}, \dots, g_{\beta_l})\left(g_{\beta_1}^{\mathrm{T}}, \dots, g_{\beta_l}^{\mathrm{T}}\right),$$

where g_{β_j} is the β_jth row of the matrix g and

$$D_2(0,a) = \left\{\left(x_{\beta_{l+1}}, \dots, x_{\beta_d}\right) \;\middle|\; \left(x_{\beta_{l+1}}^2 + \dots + x_{\beta_d}^2\right)^{1/2} \le a\right\}.$$

We construct the set

$$H(\beta_1, \dots, \beta_l) = \Big\{(t, x_{\beta_1}, \dots, x_{\beta_l}) \;\Big|$$

for any open neighborhood[1] $U(t, x_{\beta_1}, \dots, x_{\beta_l})$ of the point $(t, x_{\beta_1}, \dots, x_{\beta_l})$, there exists an $a > 0$ such that the integral

$$\int\limits_{U(t,x_{\beta_1},\dots,x_{\beta_l})} \sup_{(x_{\beta_{l+1}},\dots,x_{\beta_d})\in D_2(0,a)} \left(\det \sigma_{\beta_1,\dots,\beta_l}(t, x_1, \dots, x_d)\right)^{-1} dt\, dx_{\beta_1} \dots dx_{\beta_l}$$

is either undefined or equal to $\infty\Big\}$.

We say that a real function $h(t,X) = h(t, x_1, \dots, x_d)$ satisfies *condition A* if there exist rows $g_{\beta_1}, \dots, g_{\beta_l}$ of the matrix g such that, for any fixed $(t, x_{\beta_1}, \dots, x_{\beta_l})$, the function h is continuous with respect to the remaining components $\left(x_{\beta_{l+1}}, \dots, x_{\beta_d}\right)$ of the vector X and the set

$$\{(t, x_1, \dots, x_d) \mid (t, x_{\beta_1}, \dots, x_{\beta_l}) \in H(\beta_1, \dots, \beta_l)\}$$

is contained in the set of points of continuity of the mapping h.

[1] An open neighborhood is treated as a neighborhood open in the space of the variables $(t, x_{\beta_1}, \dots, x_{\beta_l})$.

A function $h : R_+ \times R^d \to R^{d\times r}$ is said to be *locally bounded* if, for each $b > 0$, there exists a constant $N(b)$ such that $\|h(t,X)\| \le N(b)$ for all $t \in [0,b]$ and $X \in B(0,b)$.

Let $g^{(1)}$ be the $l \times d$ matrix consisting of the first l rows of the matrix g, let $g^{(2)}$ be the $(d-l)\times d$ matrix consisting of the remaining rows of the matrix g, let $f^{(1)}$ be the vector consisting of the first l components of the vector f, and let $f^{(2)}$ be the vector consisting of the remaining components of the vector f. Next, let $X = (x,y)$, $x \in R^l$, $y \in R^{d-l}$, $\sigma^{(1)} = g^{(1)}g^{(1)\mathrm{T}}$, $B_1(0,a) = \{x \in R^l \mid \|x\| \le a\}$, $B_2(0,a) = \{y \in R^{d-l} \mid \|y\| \le a\}$, $H = \Big\{(t,x) \in R_+ \times R^l \mid$ for any open neighborhood $U(t,x)$ of the point (t,x) in $R_+ \times R^l$, there exists a number $a > 0$ such that the integral

$$\int\limits_{U(t,x)} \sup_{y\in B_2(0,a)} \left(\det \sigma^{(1)}(\tau,z,y)\right)^{-1} d\tau\, dz$$

is either indefinite or equal to $\infty\Big\}$, and

$$H^c = \left(R_+ \times R^l\right)\backslash H, \qquad (H)_\gamma = \left\{(t,x) \in R_+ \times R^l \mid \sup_{(s,y)\in H} (|t-s| + \|x-y\|) < \gamma\right\},$$
$$(H)^c_\gamma = \left(R_+ \times R^l\right)\backslash (H)_\gamma.$$

Now we consider the system of the form (2) with the above-constructed functions $f^{(1)}$, $f^{(2)}$, $g^{(1)}$, and $g^{(2)}$.

Lemma 1. *Let* $(\Omega, \mathscr{F}, P, \mathscr{F}_t, W(t), x(t), y(t),\ t \in R_+)$ *be a weak solution of system* (2), *and let the functions* $f^{(1)}$, $f^{(2)}$, $g^{(1)}$, *and* $g^{(2)}$ *be locally bounded and Borel measurable. Then for arbitrary* $a > 0$ *and* $T > 0$, *there exists a constant* $c(a,T,l,d)$ *such that*

$$E\left(\int\limits_0^{T\wedge\tau^a} \left(\det \sigma^{(1)}(t,x(t),y(t))\right)^{1/(l+1)} \psi(t,x(t),y(t))dt\right)$$
$$\le c(a,T,l,d)\left(\int\limits_{[0,T]\times B_1(0,a)} \sup_{y\in B_2(0,a)} \psi^{l+1}(t,x,y)dt\,dx\right)^{1/(l+1)}, \tag{3}$$

where $\tau^a = \inf\{t \mid \|x(t)\| \vee \|y(t)\| > a\}$, *for any nonnegative Borel measurable function* $\psi(t,x,y)$ *such that the mapping* $(t,x) \to \sup_{y\in B_2(0,b)} \psi(t,x,y)$ *is Lebesgue measurable for each* $b > 0$.

Proof. Take arbitrary $T > 0$ and $a > 0$. Let $q : R_+ \times R^l \to R_+$ be a bounded continuous function. We set $q(t,x) = 0$ for $t < 0$. By the Krylov lemma [2, Lemma II.2.7], there exists a bounded function $z(t,x) \le 0$ vanishing for $t < 0$ and such that the following conditions are satisfied for all sufficiently large n and for all $(t,x) \in R_+ \times B_1(0,a)$.

1.

$$c_1(a,l)\left(\det \sigma^{(1)}(T-t,x,y)\right)^{1/(l+1)} q_n(t,x) \le -\frac{\partial z_n(t,x)}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{l} \sigma^{(1)ij}(T-t,x,y)\frac{\partial^2 z_n(t,x)}{\partial x^i \partial x^j},$$

where $c_1(a,l)$ is a positive constant, $\sigma^{(1)ij}$ are the entries of the matrix $\sigma^{(1)}$,

$$\sigma^{(1)ij}(T-t,x,y) = 0 \quad \text{for} \quad t > T,$$

$z_n(t,x)$ is the convolution of the functions $z(t,x)$ and $J_n(t,x)$, i.e.,

$$z_n(t,x) = z(t,x) * J_n(t,x) = \int\limits_{|t-\tau|\le 1/n,\, \|x-\eta\|\le 1/n} z(\tau,\eta) J_n(t-\tau, x-\eta) d\tau\, d\eta,$$

$$q_n(t,x) = q(t,x) * J_n(t,x), \qquad J_n(t,x) = n^{l+1}\zeta(nt, nx),$$

and $\zeta(t,x)$ is a nonnegative infinitely differentiable function vanishing for $\|x\| > 1$ and $|t| > 1$ and satisfying $\int_{|t|\le 1} dt \int_{\|x\|\le 1} \zeta(t,x)dx = 1$.

2. If $b \in R^l$ and $c > 0$ satisfy the condition $\|b\| \le ac/2$, then

$$-\sum_{i=1}^{l} \frac{\partial z_n(t,x)}{\partial x^i} b_i \le c\,|z_n(t,x)|$$

for all $(t,x) \in R_+ \times B_1(0,a)$.

3. There exists a constant $c_2(a,l)$ such that

$$|z(t,x)| \le c_2(a,l) \left(\int\limits_{[0,t]\times B_1(0,a)} q^{l+1}(s,x) ds\, dx \right)^{1/(l+1)}$$

for all $(t,x) \in R_+ \times R^l$.

We set

$$I(q_n) = \int\limits_0^{T\wedge\tau^a} \left(\det \sigma^{(1)}(t, x(t), y(t))\right)^{1/(l+1)} q_n(T-t, x(t)) dt.$$

By using the Itô formula and relations 1–3, we obtain

$$\begin{aligned}
E\left(I(q_n)\right) &\le \frac{1}{c_1} E\left(\int\limits_0^{T\wedge\tau^a} \left(-\frac{\partial z_n(T-t, x(t))}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{l} \sigma^{(1)ij}(t, x(t), y(t)) \frac{\partial^2 z_n(T-t, x(t))}{\partial x^i \partial x^j} \right) dt \right) \\
&= \frac{1}{c_1} E\Bigg(z_n\left(T - (T\wedge\tau^a), x(T\wedge\tau^a)\right) - z_n(T, x(0)) \\
&\quad - \int\limits_0^{T\wedge\tau^a} \sum_{i=1}^{l}\sum_{j=1}^{d} \frac{\partial z_n(T-t, x(t))}{\partial x^i} g^{(1)ij}(t, x(t), y(t)) dW^j(t) \\
&\quad - \int\limits_0^{T\wedge\tau^a} \sum_{i=1}^{l} \frac{\partial z_n(T-t, x(t))}{\partial x^i} f^{(1)i}(t, x(t), y(t)) dt \Bigg) \\
&\le c_3(a,T,l,d) \sup_{0\le t\le T,\, x\in B_1(0,a)} |z_n(t,x)| \le c_3(a,T,l,d) \sup_{0\le t\le T,\, x\in B_1(0,a)} |z(t,x)| \\
&\le c_4(a,T,l,d) \left(\int\limits_{[0,T]\times B_1(0,a)} q^{l+1}(t,x) dt\, dx \right)^{1/(l+1)}. \qquad (4)
\end{aligned}$$

Let $q_n(T-t,x) = r_n(t,x)$ and $q(T-t,x) = r(t,x)$; from inequality (4) and the Fatou lemma, we have

$$c_4 \left(\int\limits_{[0,T]\times B_1(0,a)} r^{l+1}(t,x) dt\, dx \right)^{1/(l+1)} = c_4 \left(\int\limits_{[0,T]\times B_1(0,a)} q^{l+1}(t,x) dt\, dx \right)^{1/(l+1)}$$

$$\geq E\left(\int_0^{T\wedge\tau^a}\left(\det\sigma^{(1)}(t,x(t),y(t))\right)^{1/(l+1)}\liminf_{n\to\infty} q_n(T-t,x(t))dt\right)$$

$$\geq E\left(\int_0^{T\wedge\tau^a}\left(\det\sigma^{(1)}(t,x(t),y(t))\right)^{1/(l+1)} q(T-t,x(t))dt\right)$$

$$= E\left(\int_0^{T\wedge\tau^a}\left(\det\sigma^{(1)}(t,x(t),y(t))\right)^{1/(l+1)} r(t,x(t))dt\right). \quad (5)$$

The last relation is valid for all nonnegative continuous bounded functions $r(t,x)$. By using Theorem I.20 in [7], we find that inequality (5) remains valid for nonnegative Lebesgue measurable bounded functions $r(t,x)$. By approximating the function $r(t,x)$ by the sequence of functions $r\wedge n$, $n\geq 1$, we obtain inequality (5) for a Lebesgue measurable nonnegative function $r(t,x)$.

Let $\psi(t,x,y)$ be an arbitrary function satisfying the assumptions of Lemma 1. Then, by applying the above-proved assertion to the function $r(t,x)=\sup_{y\in B_2(0,a)}\psi(t,x,y)$, we obtain

$$E\left(\int_0^{T\wedge\tau^a}\left(\det\sigma^{(1)}(t,x(t),y(t))\right)^{1/(l+1)}\psi(t,x(t),y(t))dt\right)$$

$$\leq E\left(\int_0^{T\wedge\tau^a}\left(\det\sigma^{(1)}(t,x(t),y(t))\right)^{1/(l+1)}\sup_{y\in B_2(0,a)}\psi(t,x(t),y)dt\right)$$

$$\leq c(a,T,l,d)\left(\int_{[0,T]\times B_1(0,a)}\sup_{y\in B_2(0,a)}\psi^{l+1}(t,x,y)dt\,dx\right)^{1/(l+1)}.$$

The proof of the lemma is complete.

Corollary 1. *Let the assumptions of Lemma* 1 *be valid, and let* $\psi(t,x,y)$ *be a nonnegative Borel measurable function continuous with respect to* y *for any* $(t,x)\in R_+\times R^l$. *Then for arbitrary* $T\in R_+$ *and* $a\in R_+$, *there exists a constant* $c(a,T,l,d)$ *such that*

$$E\left(\int_0^{T\wedge\tau^a} 1_{(H)^c_\epsilon}(t,x(t))\psi(t,x(t),y(t))dt\right)$$

$$\leq c(a,T,l,d)\left(\int_{([0,T]\times B_1(0,a))\cap(H)^c_\epsilon}\sup_{y\in B_2(0,a)}\left(\det\sigma^{(1)}(t,x,y)\right)^{-1}\sup_{y\in B_2(0,a)}\psi^{l+1}(t,x,y)dt\,dx\right)^{1/(l+1)}$$

for each $\epsilon>0$, *where* $\tau^a=\inf\{t\mid \|x(t)\|\vee\|y(t)\|>a\}$.

Indeed, since the integrability and hence the Lebesgue measurability of the function

$$(t,x)\to 1_{(H)^c_\epsilon}(t,x)\sup_{y\in B_2(0,a)}\left(\det\sigma^{(1)}(t,x,y)\right)^{-1/(l+1)},\quad (t,x)\in[0,T]\times B_1(0,a),$$

follow from the definition of the set H, from Corollary 1, we find that it suffices to apply Lemma 1 to the function $\psi_1(t,x,y)=1_{(H)^c_\epsilon}(t,x)\left(\det\sigma^{(1)}(t,x,y)\right)^{-1/(l+1)}\sup_{y\in B_2(0,a)}\psi(t,x,y)$. [We assume that $\psi_1(t,x,y)=0$ if $1_{(H)^c_\epsilon}(t,x)=0$.]

Consider the matrices $\sigma_n = T\Lambda_n T^{\mathrm{T}}$, where

$$\Lambda_n = \operatorname{diag}\left((\lambda_1 + 1/n) \wedge n, \ldots, (\lambda_d + 1/n) \wedge n\right),$$
$$g_n = T \operatorname{diag}\left(((\lambda_1 + 1/n) \wedge n)^{1/2}, \ldots, ((\lambda_d + 1/n) \wedge n)^{1/2}\right),$$
$$f_n(t,X) = \left(f_n^i(t,X)\right), \qquad f_n^i(t,X) = \left(f^i(t,X) \vee (-n)\right) \wedge n, \qquad i = 1,\ldots,d, \qquad n \in N.$$

We divide the matrices g_n and f_n into submatrices $g_n^{(1)}$, $g_n^{(2)}$, $f_n^{(1)}$, and $f_n^{(2)}$ in the same way as the matrices g and f have been divided into the submatrices $g^{(1)}$, $g^{(2)}$, $f^{(1)}$, and $f^{(2)}$. For each positive integer n, there exists a constant $\alpha_n > 0$ such that $\det g_n g_n^{\mathrm{T}} = \det \sigma_n \ge \alpha_n$ for all $(t,X) \in R_+ \times R^d$; moreover, $\lim_{n\to\infty} f_n(t,X) = f(t,X)$ and $\lim_{n\to\infty} \sigma_n(t,X) = \sigma(t,X)$ at each point $(t,X) \in R_+ \times R^d$.

Corollary 2. *Let $a \in R_+$ and $T \in R_+$. Let f and g be locally bounded Borel measurable functions. Let $X_n(t) = (x_n(t), y_n(t))$ be a sequence of weak solutions of the systems*

$$dx(t) = f_n^{(1)}(t,x(t),y(t))dt + g_n^{(1)}(t,x(t),y(t))dW(t),$$
$$dy(t) = f_n^{(2)}(t,x(t),y(t))dt + g_n^{(2)}(t,x(t),y(t))dW(t).$$

Let $\left(\hat{X}_n(t)\right)$, $n \ge 1$, be a sequence of continuous processes such that $P^{(\hat{X}_n,\hat{\tau}_n^a)} = P^{(X_n,\tau_n^a)}$ and $\hat{X}_n(s) \to_{n\to\infty} \hat{X}(s) = (\hat{x}(s),\hat{y}(s))$ uniformly on each closed interval in R_+ a.s., $\hat{\tau}_n^a \to_{n\to\infty} \hat{\tau}^a$ a.s., where $\hat{\tau}_n^a$, $\hat{\tau}^a$, and τ_n^a are stopping times such that

$$\|x_n(t)\| \vee \|y_n(t)\| \le a \qquad \forall t \le \tau_n^a,$$
$$\|\hat{x}_n(t)\| \vee \|\hat{y}_n(t)\| \le a \qquad \forall t \le \hat{\tau}_n^a,$$
$$\|\hat{x}(t)\| \vee \|\hat{y}(t)\| \le a \qquad \forall t \le \hat{\tau}^a.$$

Then

$$E\left(\int_0^{T\wedge\hat{\tau}^a} 1_{(H)_\epsilon^c}\left(t,\hat{x}(t)\right)\psi\left(t,\hat{x}(t),\hat{y}(t)\right)dt\right) \le c(a,T,l,d)$$
$$\times\left(\int\limits_{([0,T]\times B_1(0,a))\cap(H)_{\epsilon/2}^c} \sup_{y\in B_2(0,a)}\left(\det\sigma^{(1)}(t,x,y)\right)^{-1}\sup_{y\in B_2(0,a)}\psi^{l+1}(t,x,y)dt\,dx\right)^{1/(l+1)} \tag{6}$$

for any $\epsilon > 0$ and any nonnegative Borel measurable function $\psi(t,x,y)$ continuous with respect to y, where $c(a,T,l,d)$ is the same constant as in Lemma 1.

Proof. Let $\epsilon > 0$. By Corollary 1, the inequality

$$E\left(\int_0^{T\wedge\tau_n^a} 1_{(H)_{\epsilon/2}^c}\left(t,x_n(t)\right)r\left(t,x_n(t)\right)dt\right) \le c(a,T,l,d)$$
$$\times\left(\int\limits_{([0,T]\times B_1(0,a))\cap(H)_{\epsilon/2}^c}\left(\sup_{y\in B_2(0,a)}\left(\det\sigma_n^{(1)}(t,x,y)\right)^{-1}\right)r^{l+1}(t,x)dt\,dx\right)^{1/(l+1)} \tag{7}$$

holds for any nonnegative continuous bounded function $r(t,x)$.

By using inequality (7), the Fatou lemma, the inequality $1_{(H)^c_{\epsilon/2}}(t,\hat{x}_n(t)) \geq 1_{(H)^c_\epsilon}(t,\hat{x}(t))$ valid for all sufficiently large n and for all $t \in [0,T]$, and the inequality $\det \sigma_n^{(1)}(t,x,y) \geq \det \sigma^{(1)}(t,x,y)$ valid for all $(t,x,y) \in [0,T] \times B_1(0,a) \times B_2(0,a)$ and for all sufficiently large n, we obtain

$$c(a,T,l,d)\left(\int\limits_{([0,T]\times B_1(0,a))\cap(H)^c_{\epsilon/2}}\left(\sup_{y\in B_2(0,a)}\left(\det\sigma^{(1)}(t,x,y)\right)^{-1}\right)r^{l+1}(t,x)dt\,dx\right)^{1/(l+1)}$$
$$\geq \liminf_{n\to\infty} c(a,T,l,d)\left(\int\limits_{([0,T]\times B_1(0,a))\cap(H)^c_{\epsilon/2}}\left(\sup_{y\in B_2(0,a)}\left(\det\sigma_n^{(1)}(t,x,y)\right)^{-1}\right)r^{l+1}(t,x)dt\,dx\right)^{1/(l+1)}$$
$$\geq \liminf_{n\to\infty} E\left(\int\limits_0^{T\wedge\tau_n^a} 1_{(H)^c_{\epsilon/2}}(t,x_n(t))\,r(t,x_n(t))\,dt\right)$$
$$= \liminf_{n\to\infty} E\left(\int\limits_0^{T\wedge\hat{\tau}_n^a} 1_{(H)^c_{\epsilon/2}}(t,\hat{x}_n(t))\,r(t,\hat{x}_n(t))\,dt\right)$$
$$\geq \liminf_{n\to\infty} E\left(\int\limits_0^{T\wedge\hat{\tau}^a} 1_{(H)^c_{\epsilon}}(t,\hat{x}(t))\,r(t,\hat{x}_n(t))\,dt\right)$$
$$\geq E\left(\int\limits_0^{T\wedge\hat{\tau}^a} 1_{(H)^c_{\epsilon}}(t,\hat{x}(t))\liminf_{n\to\infty} r(t,\hat{x}_n(t))\,dt\right) = E\left(\int\limits_0^{T\wedge\hat{\tau}^a} 1_{(H)^c_{\epsilon}}(t,\hat{x}(t))\,r(t,\hat{x}(t))\,dt\right).$$

It follows from the theorem on monotone classes that the last inequality remains valid for arbitrary Lebesgue measurable nonnegative functions $r(t,x)$. By applying this inequality to the function $r(t,x) = \sup_{y\in B_2(0,a)} \psi(t,x,y)$ and by following the lines of the proof of Lemma 1, we obtain the desired inequality (6).

Lemma 2. *Let $f(t,x,y)$ be a real Borel measurable locally bounded function continuous with respect to y, and let $f_n(t,x,y) = f(t,x,y) * J_n(t,x)$, $n \geq 1$. Then the convergence*

$$\int\limits_{([0,T]\times B_1(0,a))\cap(H)^c_\gamma} \sup_{y\in B_2(0,a)}\left(\det\sigma^{(1)}(t,x,y)\right)^{-1} \sup_{y\in B_2(0,a)} \left|f_n(t,x,y)-f(t,x,y)\right|^{l+1} dt\,dx \underset{n\to\infty}{\to} 0$$

takes place for arbitrary $a \in R_+$, $T \in R_+$, and $\gamma > 0$.

Proof. Take $\epsilon > 0$, $a \in R_+$, and $T \in R_+$. Let

$$\tilde{D} = ([0,T]\times B_1(0,a)) \cap (H)^c_\gamma, \qquad D_1 = ([-1,T+1]\times B_1(0,a+1)) \cap (H)^c_\gamma;$$

then $\int_{D_1} \sup_{y\in B_2(0,a)} \left(\det\sigma^{(1)}(t,x,y)\right)^{-1} dt\,dx < \infty$. There exists a $\delta(\epsilon) > 0$ such that

$$\int\limits_E \sup_{y\in B_2(0,a)} \left(\det\sigma^{(1)}(t,x,y)\right)^{-1} dt\,dx \leq \epsilon \tag{8}$$

for any set $E \subset D_1$ with $\mu(E) \leq \delta(\epsilon)$ (where μ is the Lebesgue measure).

By the Scorza-Dragoni theorem [8], there exists a closed set

$$K(a,T,\delta(\epsilon)) \subset [-1,T+1]\times B_1(0,a+1)$$

such that the restriction of the function f to $K \times B_2(0, a+1)$ is continuous and

$$\mu\left(\left([-1, T+1] \times B_1(0, a+1)\right) \backslash K\right) \le \delta(\epsilon).$$

By the Cantor theorem, there exists a $\nu(\epsilon, a, T)$ such that $|f(t_1, x_1, y_1) - f(t_2, x_2, y_2)| \le \epsilon$ for arbitrary

$$\begin{aligned}&(t_1, x_1, y_1), (t_2, x_2, y_2) \in K \times B_2(0, a+1),\\ &|t_2 - t_1| \le \nu(\epsilon, a, T), \qquad \|x_2 - x_1\| \le \nu(\epsilon, a, T), \qquad \|y_2 - y_1\| \le \nu(\epsilon, a, T).\end{aligned}$$

It follows that

$$\sup_{y_1, y_2 \in B_2(0,a),\, \|y_1 - y_2\| \le \nu(\epsilon, a, T)} |f(t-\tau, x-z, y_1) - f(t-\tau, x-z, y_2)| \le \epsilon \tag{9}$$

for arbitrary

$$(t, x) \in K \cap ([0, T] \times B_1(0, a)),$$

arbitrary τ with $|\tau| \le 1$, and arbitrary z with $\|z\| \le 1$.

Now it follows from (8) and (9) that

$$\begin{aligned}&\int\limits_{\tilde D} \sup_{y \in B_2(0,a)} \left(\det \sigma^{(1)}(t, x, y)\right)^{-1}\\ &\quad\times \sup_{\substack{y_1, y_2 \in B_2(0,a)\\ \|y_1 - y_2\| \le \nu(\epsilon, a, T)}} |f(t-\tau, x-z, y_1) - f(t-\tau, x-z, y_2)|^{l+1}\, dt\, dx\\ &= \int\limits_{\tilde D \cap K} \sup_{y \in B_2(0,a)} \left(\det \sigma^{(1)}(t, x, y)\right)^{-1}\\ &\quad\times \sup_{\substack{y_1, y_2 \in B_2(0,a)\\ \|y_1 - y_2\| \le \nu(\epsilon, a, T)}} |f(t-\tau, x-z, y_1) - f(t-\tau, x-z, y_2)|^{l+1}\, dt\, dx\\ &\quad+ \int\limits_{\tilde D \backslash K} \sup_{\substack{y_1, y_2 \in B_2(0,a)\\ \|y_1 - y_2\| \le \nu(\epsilon, a, T)}} |f(t-\tau, x-z, y_1) - f(t-\tau, x-z, y_2)|^{l+1}\\ &\quad\times \sup_{y \in B_2(0,a)} \left(\det \sigma^{(1)}(t, x, y)\right)^{-1} dt\, dx \le C\epsilon^{l+1}\end{aligned} \tag{10}$$

for all τ and z, $|\tau| \le 1$, $\|z\| \le 1$.

By using inequalities (9) and (10) and the generalized Minkowski inequality, we obtain

$$\begin{aligned}&\left(\int\limits_{\tilde D} \sup_{y \in B_2(0,a)} \left(\det \sigma^{(1)}(t, x, y)\right)^{-1} \sup_{\substack{y_1, y_2 \in B_2(0,a)\\ \|y_1 - y_2\| \le \nu(\epsilon, a, T)}} |f_n(t, x, y_1) - f_n(t, x, y_2)|^{l+1}\, dt\, dx\right)^{1/(l+1)}\\ &\quad\le \left(\int\limits_{\tilde D} dt\, dx \left(\int\limits_{\substack{|\tau| \le 1/n\\ \|z\| \le 1/n}} \sup_{y \in B_2(0,a)} \left(\det \sigma^{(1)}(t, x, y)\right)^{-1/(l+1)}\right.\right.\\ &\qquad\left.\left.\times \sup_{\substack{y_1, y_2 \in B_2(0,a)\\ \|y_1 - y_2\| \le \nu(\epsilon, a, T)}} |f(t-\tau, x-z, y_1) - f(t-\tau, x-z, y_2)|\, J_n(\tau, z) d\tau\, dz\right)^{l+1}\right)^{1/(l+1)}\end{aligned}$$

$$\leq \int\limits_{\substack{|\tau|\leq 1/n \\ \|z\|\leq 1/n}} d\tau\, dz \left(\int\limits_{\tilde{D}} \sup_{y\in B_2(0,a)} \left(\det \sigma^{(1)}(t,x,y)\right)^{-1} \right.$$

$$\left. \times \sup_{\substack{y_1,y_2\in B_2(0,a) \\ \|y_1-y_2\|\leq \nu(\epsilon,a,T)}} \left|f\left(t-\tau,x-z,y_1\right)-f\left(t-\tau,x-z,y_2\right)\right|^{l+1} J_n^{l+1}(\tau,z)dt\, dx \right)^{1/(l+1)}$$

$$\leq \int\limits_{\substack{|\tau|\leq 1/n \\ \|z\|\leq 1/n}} C^{1/(l+1)}\epsilon J_n(\tau,z)d\tau\, dz \leq C_1\epsilon. \tag{11}$$

The relation

$$\int\limits_{\tilde{D}} \sup_{y\in B_2(0,a)} \left(\det \sigma^{(1)}(t,x,y)\right)^{-1} \left|f_n(t,x,y)-f(t,x,y)\right|^{l+1} dt\, dx \underset{n\to\infty}{\to} 0$$

is valid for each $y \in B_2(0,a)$.

Let $Y = \{y_k\}$ be a finite $\nu(\epsilon,a,T)$-net for $B_2(0,a)$. There exists an $n_0(\epsilon)$ such that

$$\int\limits_{\tilde{D}} \sup_{y\in B_2(0,a)} \left(\det \sigma^{(1)}(t,x,y)\right)^{-1} \sup_{y_k\in Y} \left|f_n\left(t,x,y_k\right)-f\left(t,x,y_k\right)\right|^{l+1} dt\, dx \leq \epsilon^{\,l+1} \tag{12}$$

for all $n \geq n_0(\epsilon)$.

By using inequalities (10)–(12) for all $n \geq n_0(\epsilon)$, we obtain the relations

$$\int\limits_{\tilde{D}} \sup_{y\in B_2(0,a)} \left(\det \sigma^{(1)}(t,x,y)\right)^{-1} \sup_{y\in B_2(0,a)} \left|f_n(t,x,y)-f(t,x,y)\right|^{l+1} dt\, dx$$

$$\leq \int\limits_{\tilde{D}} \sup_{y\in B_2(0,a)} \left(\det \sigma^{(1)}(t,x,y)\right)^{-1} \sup_{\substack{y\in B_2(0,a) \\ y_k\in Y \\ \|y-y_k\|\leq \nu(\epsilon,a,T)}} \left|f_n(t,x,y)-f_n\left(t,x,y_k\right)\right|^{l+1} dt\, dx$$

$$+ \int\limits_{\tilde{D}} \sup_{y\in B_2(0,a)} \left(\det \sigma^{(1)}(t,x,y)\right)^{-1} \sup_{y_k\in Y} \left|f_n\left(t,x,y_k\right)-f\left(t,x,y_k\right)\right|^{l+1} dt\, dx$$

$$+ \int\limits_{\tilde{D}} \sup_{y\in B_2(0,a)} \left(\det \sigma^{(1)}(t,x,y)\right)^{-1} \sup_{\substack{y\in B_2(0,a) \\ y_k\in Y \\ \|y-y_k\|\leq \nu(\epsilon,a,T)}} \left|f(t,x,y)-f\left(t,x,y_k\right)\right|^{l+1} dt\, dx \leq C_2\epsilon^{\,l+1}.$$

The proof of Lemma 2 is complete.

Theorem. *Let $f(t,X)$ and $g(t,X)$ be Borel measurable locally bounded functions, and let the components of the functions $f(t,X)$ and $\sigma(t,X) = g(t,X)g^{\mathrm{T}}(t,X)$ satisfy condition* A. *Then for any given probability ν on $\left(R^d, \mathscr{B}\left(R^d\right)\right)$, Eq.* (1) *has a weak solution with the initial distribution ν.*

Proof. By the Krylov theorem [2, Th. II.6.1], for each $n \in N$, the equation

$$X_n(t) = X_n(0) + \int\limits_0^t f_n\left(\tau, X_n(\tau)\right) d\tau + \int\limits_0^t g_n\left(\tau, X_n(\tau)\right) dW_n(\tau), \qquad t \in R_+, \tag{13}$$

has a weak solution $(\Omega_n, \mathscr{F}_n, P_n, \mathscr{F}_{nt}, W_n(t), X_n(t),\ t \in R_+)$ with the initial distribution ν.

We set $\tau_n^m = \inf\{t \mid \|X_n(t)\| > m\}$ and $X_n^m(t) = X_n(t \wedge \tau_n^m)$ and consider the double sequence

$$\begin{pmatrix} (X_1^1, \tau_1^1) & (X_1^2, \tau_1^2) & \dots & (X_1^m, \tau_1^m) & \dots \\ (X_2^1, \tau_2^1) & (X_2^2, \tau_2^2) & \dots & (X_2^m, \tau_2^m) & \dots \\ \dots & \dots & \dots & \dots & \dots \\ (X_n^1, \tau_n^1) & (X_n^2, \tau_n^2) & \dots & (X_n^m, \tau_n^m) & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Set $\Psi_k = ((X_k^1, \tau_k^1), (X_k^2, \tau_k^2), \dots, (X_k^m, \tau_k^m), \dots)$, $k = 1, 2, \dots$

We introduce a metric ϱ in $\left(C\left([0,+\infty), R^d\right), [0,+\infty]\right)$ and a metric D in

$$\left(\left(C\left([0,+\infty), R^d\right), [0,+\infty]\right) \times \dots \times \left(C\left([0,+\infty), R^d\right), [0,+\infty]\right) \times \cdots\right)$$

as follows:

$$\varrho\left((z,\tau), \left(z^1, \tau^1\right)\right) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left(\sup_{0 \le t \le n} \left\|z(t) - z^1(t)\right\| \wedge 1\right) + \left|\frac{\tau}{1+\tau} - \frac{\tau^1}{1+\tau^1}\right|,$$

$$D\left(\left(\left(X_n^1, \tau_n^1\right), \dots, \left(X_n^m, \tau_n^m\right), \dots\right), \left(\left(X_k^1, \tau_k^1\right), \dots, \left(X_k^m, \tau_k^m\right), \dots\right)\right) = \sum_{m=1}^{\infty} \frac{1}{2^{m+1}} \varrho\left(\left(X_n^m, \tau_n^m\right), \left(X_k^m, \tau_k^m\right)\right).$$

For any $T > 0$ and any fixed $m \in N$, there exist constants $M_1(m)$ an $M(m,T)$ such that the following relations are valid.

1. $\sup_n E\left(\|X_n^m(0)\|^2\right) \le M_1(m)$.
2. $\sup_n E\left(\|X_n^m(t) - X_n^m(s)\|^4\right) \le M(m,T)|t-s|^2$ for arbitrary $s, t \in [0,T]$.

It follows from Theorem I.4.3 in [6] that the sequence

$$\left(X_n^m, \tau_n^m\right), \qquad n \ge 1,$$

is dense in $\left(C\left([0,+\infty), R^d\right), [0,+\infty]\right)$ for each $m \in N$.

Lemma 3. *The sequence* Ψ_n, $n \ge 1$, *is dense in the space*

$$\left(\left(C\left([0,+\infty), R^d\right), [0,+\infty]\right) \times \dots \times \left(C\left([0,+\infty), R^d\right), [0,+\infty]\right) \times \cdots\right).$$

Proof. Take an arbitrary $\epsilon > 0$. For any positive integer m, there exists a compact set $K_m \in \left(C\left([0,+\infty), R^d\right), [0,+\infty]\right)$ such that $P^{(X_n^m, \tau_n^m)}(K_m) \ge 1 - \epsilon/2^m$ for all $n \in N$. Let $K = K_1 \times \dots \times K_m \times \cdots$ Let us show that K is a compact set in the space

$$\left(\left(C\left([0,+\infty), R^d\right), [0,+\infty]\right) \times \dots \times \left(C\left([0,+\infty), R^d\right), [0,+\infty]\right) \times \cdots\right).$$

For any $\delta > 0$, we take an $m = m(\delta)$ such that $1/2^m < \delta/2$. For each K_j, $j = 1, \dots, m$, there exists a finite $(\delta/2)$-net $\{s_1^j, \dots, s_{n_j}^j\}$. For K_j, $j \ge m+1$, we take an arbitrary element $s^j \in K_j$.

Let

$$S = \left\{\left(s_{k_1}^1, \dots, s_{k_m}^m, s^{m+1}, s^{m+2}, \dots\right) \mid k_1 \in \{1, \dots, n_1\}, \ \dots, \ k_m \in \{1, \dots, n_m\}\right\}.$$

For each $\tilde{k} \in K$, there exists an $\tilde{s} \in S$ such that

$$\tilde{D}\left(\tilde{k}, \tilde{s}\right) \le \sum_{k=1}^{m} \frac{1}{2^k} \frac{\delta}{2} + \sum_{k=m+1}^{\infty} \frac{1}{2^k} < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Consequently, S is a finite δ-net for K. Obviously, K is a closed set. Therefore, K is compact. Since the sequence (X_n^m, τ_n^m), $n \geq 1$, is dense in $\left(C\left([0,+\infty), R^d\right), [0,+\infty]\right)$ for each $m \in N$, we obtain

$$P^{\Psi_n}(K) \geq 1 - \epsilon \sum_{m=1}^{\infty} \frac{1}{2^m} = 1 - \epsilon.$$

The proof of the lemma is complete.

The sequence Ψ_n, $n \geq 1$, satisfies the assumptions of the Skorokhod theorem [6, Th. I.2.7]. Its proof implies that there exists a subsequence n_k of the sequence n (to simplify the notation, we write n instead of n_k) and processes

$$\varepsilon_n = \left(\left(z_n^1, \eta_n^1\right), \ldots, \left(z_n^m, \eta_n^m\right), \ldots\right), \qquad \varepsilon = \left(\left(z^1, \eta^1\right), \ldots, \left(z^m, \eta^m\right), \ldots\right)$$

on some probability space $(\Omega, \mathscr{F}, P)$ such that the processes $z_n^m(t)$ and $z^m(t)$ are continuous, $P^{\varepsilon_n} = P^{\Psi_n}$, $z_n^m(t) \to_{n\to\infty} z^m(t)$ uniformly on each compact set in R_+ a.s., and $\eta_n^m \to_{n\to\infty} \eta^m$ a.s. In addition, $z^m(t) = z^{m+1}(t)$ for $t < \eta^m$, and $\eta^m \leq \eta^{m+1}$ a.s. Let $e = \lim_{m\to\infty} \eta^m$. We define a process $z(t)$ as follows: $z(t) = z^m(t)$ for $t \leq \eta^m$, $\eta^m < \infty$, $z(t) = z^m(t)$ for $t < \eta^m$, $\eta^m = \infty$, and $z(t) = 0$ for $t \geq e$. By $\sigma_{t+\epsilon}$ we denote the minimum σ-algebra with respect to which all random vectors $z^m(s)$, $0 \leq s \leq t+\epsilon$, $m \geq 1$, are measurable. Let $\mathscr{F}_t = \bigcap_{\epsilon>0} \sigma_{t+\epsilon}$; then the process $z(t)1_{[0,e)}(t)$ is $(\mathscr{F}_t)$-coordinated and has continuous trajectories for $t < e$. Moreover, e is a $(\mathscr{F}_t)$-stopping moment and $\limsup_{t\uparrow e} \|z(t)\| = \infty$ for $e < \infty$.

We fix an $m \in N$ and take arbitrary $s, t \in R_+$, $s \leq t$, an arbitrary twice continuously differentiable function $h : R^d \to R$ bounded together with its partial derivatives of order ≤ 2, and an arbitrary continuous bounded $\left(\mathscr{B}_s\left(C\left(R_+, R^d\right)\right)\right)$-measurable function $q : C\left(R_+, R^d\right) \to R$.

Relation (13), together with the Itô formula, implies that

$$E_n\left(\left(h\left(X_n^m(t)\right) - h\left(X_n^m(s)\right) - \int_{s\wedge\tau_n^m}^{t\wedge\tau_n^m}\left(\frac{1}{2}\sum_{i,j=1}^{d} \sigma_n^{ij}\left(\tau, X_n^m(\tau)\right) h_{x_i x_j}\left(X_n^m(\tau)\right) + \sum_{i=1}^{d} f_n^i\left(\tau, X_n^m(\tau)\right) h_{x_i}\left(X_n^m(\tau)\right)\right) d\tau\right) q\left(X_n^m\right)\right) = 0. \tag{14}$$

We fix the component $f^i(t,X)$ of the vector f with index i. By using condition A, we take the rows of the matrix g with indices $\beta_1, \ldots, \beta_l$ such that the function $f^i(t,X)$ is continuous with respect to the variables $\hat{\hat{x}} = (x_{\beta_l+1}, \ldots, x_{\beta_d})$ for any fixed $(t, \hat{x}) = (t, x_{\beta_1}, \ldots, x_{\beta_l})$ and the set $\{(t, x_1, \ldots, x_d) \mid (t, x_{\beta_1}, \ldots, x_{\beta_l}) \in H(\beta_1, \ldots, \beta_l)\}$ is contained in the set of points of continuity of the function $f^i(t,X)$. (Without loss of generality, one can assume that $\beta_1 = 1, \ldots, \beta_l = l$.) Each of the processes X_n, X_n^m, z, z_n^m, and z^m splits into two processes, $X_n = \left(\hat{X}_n, \hat{\hat{X}}_n\right)$, $X_n^m = \left(\hat{X}_n^m, \hat{\hat{X}}_n^m\right)$, $z = \left(\hat{z}, \hat{\hat{z}}\right)$, $z_n^m = \left(\hat{z}_n^m, \hat{\hat{z}}_n^m\right)$, and $z^m = \left(\hat{z}^m, \hat{\hat{z}}^m\right)$. For simplicity, we write H instead of $H(\beta_1, \ldots, \beta_l)$ and set $(\sigma_n)_{1,\ldots,l}(t, x_1, \ldots, x_d) = a_n\left(t, \hat{x}, \hat{\hat{x}}\right)$ and $\sigma_{1,\ldots,l}(t, x_1, \ldots, x_d) = a\left(t, \hat{x}, \hat{\hat{x}}\right)$.

Take a sequence $\epsilon_k \downarrow 0$ as $k \to \infty$. Let us prove the relation

$$\lim_{n\to\infty} E\left(\left(\int_{s\wedge\eta_n^m}^{t\wedge\eta_n^m} 1_{(H)_{\epsilon_k}^c}\left(\tau, \hat{z}_n^m(\tau)\right) f_n^i\left(\tau, \hat{z}_n^m(\tau), \hat{\hat{z}}_n^m(\tau)\right) h_{x_i}\left(\hat{z}_n^m(\tau), \hat{\hat{z}}_n^m(\tau)\right) d\tau\right) q\left(\hat{z}_n^m, \hat{\hat{z}}_n^m\right)\right)$$
$$= E\left(\left(\int_{s\wedge\eta^m}^{t\wedge\eta^m} 1_{(H)_{\epsilon_k}^c}\left(\tau, \hat{z}^m(\tau)\right) f^i\left(\tau, \hat{z}^m(\tau), \hat{\hat{z}}^m(\tau)\right) h_{x_i}\left(\hat{z}^m(\tau), \hat{\hat{z}}^m(\tau)\right) d\tau\right) q\left(\hat{z}^m, \hat{\hat{z}}^m\right)\right) \equiv J. \tag{15}$$

It follows from the local boundedness of f^i and the construction of f_n^i that, to prove relation (15), it suffices to show that

$$\lim_{n\to\infty} E\left(\left(\int\limits_{s\wedge\eta_n^m}^{t\wedge\eta_n^m} 1_{(H)^c_{\epsilon_k}}\left(\tau, \hat{z}_n^m(\tau)\right) f^i\left(\tau, \hat{z}_n^m(\tau), \hat{\hat{z}}_n^m(\tau)\right) h_{x_i}\left(\hat{z}_n^m(\tau), \hat{\hat{z}}_n^m(\tau)\right) d\tau\right) q\left(\hat{z}_n^m, \hat{\hat{z}}_n^m\right)\right) = J. \tag{16}$$

Let $\tilde{f}_r^i\left(t, \hat{x}, \hat{\hat{x}}\right) = f^i\left(t, \hat{x}, \hat{\hat{x}}\right) * J_r\left(t, \hat{x}\right)$, $r \ge 1$. By using Corollary 1 and Lemma 2, we obtain the relations

$$\begin{aligned}
&\lim_{r\to\infty}\limsup_{n\to\infty} E\Bigg|\Bigg(\int\limits_{s\wedge\eta_n^m}^{t\wedge\eta_n^m} 1_{(H)^c_{\epsilon_k}}\left(\tau, \hat{z}_n^m(\tau)\right)\left(f^i\left(\tau, \hat{z}_n^m(\tau), \hat{\hat{z}}_n^m(\tau)\right) - \tilde{f}_r^i\left(\tau, \hat{z}_n^m(\tau), \hat{\hat{z}}_n^m(\tau)\right)\right)\\
&\qquad\times h_{x_i}\left(\hat{z}_n^m(\tau), \hat{\hat{z}}_n^m(\tau)\right) d\tau\Bigg) q\left(\hat{z}_n^m, \hat{\hat{z}}_n^m\right)\Bigg|\\
&\quad\le C\lim_{r\to\infty}\Bigg(\int\limits_{([0,t]\times B_1(0,m))\cap(H)^c_{\epsilon_k}} \sup_{\|\hat{\hat{x}}\|\le m}\left(\det a\left(\tau, \hat{x}, \hat{\hat{x}}\right)\right)^{-1}\\
&\qquad\times\sup_{\|\hat{\hat{x}}\|\le m}\left|f^i\left(\tau, \hat{x}, \hat{\hat{x}}\right) - \tilde{f}_r^i\left(\tau, \hat{x}, \hat{\hat{x}}\right)\right|^{l+1} d\tau\, d\hat{x}\Bigg)^{1/(l+1)} = 0.
\end{aligned} \tag{17}$$

Now, by (17), to prove relation (16), it remains to show that

$$\begin{aligned}
\lim_{r\to\infty}\lim_{n\to\infty} E\Bigg(\Bigg(&\int\limits_{s\wedge\eta_n^m}^{t\wedge\eta_n^m} 1_{(H)^c_{\epsilon_k}}\left(\tau, \hat{z}_n^m(\tau)\right)\tilde{f}_r^i\left(\tau, \hat{z}_n^m(\tau), \hat{\hat{z}}_n^m(\tau)\right)\\
&\times h_{x_i}\left(\hat{z}_n^m(\tau), \hat{\hat{z}}_n^m(\tau)\right) d\tau\Bigg) q\left(\hat{z}_n^m, \hat{\hat{z}}_n^m\right)\Bigg) = J.
\end{aligned}$$

Indeed,

$$\begin{aligned}
&\lim_{r\to\infty}\lim_{n\to\infty} E\Bigg(\Bigg(\int\limits_{s\wedge\eta_n^m}^{t\wedge\eta_n^m} 1_{(H)^c_{\epsilon_k}}\left(\tau, \hat{z}_n^m(\tau)\right)\tilde{f}_r^i\left(\tau, \hat{z}_n^m(\tau), \hat{\hat{z}}_n^m(\tau)\right)\\
&\qquad\qquad\times h_{x_i}\left(\hat{z}_n^m(\tau), \hat{\hat{z}}_n^m(\tau)\right) d\tau\Bigg) q\left(\hat{z}_n^m, \hat{\hat{z}}_n^m\right)\Bigg)\\
&= \lim_{r\to\infty}\lim_{n\to\infty} E\Bigg[\int\limits_{s\wedge\eta^m}^{t\wedge\eta^m}\Big(1_{(H)^c_{\epsilon_k}}\left(\tau, \hat{z}_n^m(\tau)\right)\tilde{f}_r^i\left(\tau, \hat{z}_n^m(\tau), \hat{\hat{z}}_n^m(\tau)\right) h_{x_i}\left(\hat{z}_n^m(\tau), \hat{\hat{z}}_n^m(\tau)\right) q\left(\hat{z}_n^m, \hat{\hat{z}}_n^m\right)\\
&\quad - 1_{(H)^c_{\epsilon_k}}\left(\tau, \hat{z}^m(\tau)\right)\tilde{f}_r^i\left(\tau, \hat{z}^m(\tau), \hat{\hat{z}}^m(\tau)\right) h_{x_i}\left(\hat{z}^m(\tau), \hat{\hat{z}}^m(\tau)\right) q\left(\hat{z}^m, \hat{\hat{z}}^m\right)\Big) d\tau\\
&\quad + \int\limits_{s\wedge\eta_n^m}^{s\wedge\eta^m}\Big(1_{(H)^c_{\epsilon_k}}\left(\tau, \hat{z}_n^m(\tau)\right)\tilde{f}_r^i\left(\tau, \hat{z}_n^m(\tau), \hat{\hat{z}}_n^m(\tau)\right) h_{x_i}\left(\hat{z}_n^m(\tau), \hat{\hat{z}}_n^m(\tau)\right) q\left(\hat{z}_n^m, \hat{\hat{z}}_n^m\right)\\
&\quad - 1_{(H)^c_{\epsilon_k}}\left(\tau, \hat{z}^m(\tau)\right)\tilde{f}_r^i\left(\tau, \hat{z}^m(\tau), \hat{\hat{z}}^m(\tau)\right) h_{x_i}\left(\hat{z}^m(\tau), \hat{\hat{z}}^m(\tau)\right) q\left(\hat{z}^m, \hat{\hat{z}}^m\right)\Big) d\tau
\end{aligned}$$

$$+\int\limits_{t\wedge\eta^m}^{t\wedge\eta_n^m}\Big(1_{(H)^c_{\epsilon_k}}(\tau,\hat{z}_n^m(\tau))\tilde{f}_r^i\left(\tau,\hat{z}_n^m(\tau),\hat{\hat{z}}_n^m(\tau)\right)h_{x_i}\left(\hat{z}_n^m(\tau),\hat{\hat{z}}_n^m(\tau)\right)q\left(\hat{z}_n^m,\hat{\hat{z}}_n^m\right)$$

$$-1_{(H)^c_{\epsilon_k}}(\tau,\hat{z}^m(\tau))\tilde{f}_r^i\left(\tau,\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right)h_{x_i}\left(\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right)q\left(\hat{z}^m,\hat{\hat{z}}^m\right)\Big)d\tau$$

$$+\int\limits_{s\wedge\eta^m}^{t\wedge\eta^m}\Big(1_{(H)^c_{\epsilon_k}}(\tau,\hat{z}^m(\tau))\left(\tilde{f}_r^i\left(\tau,\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right)-f^i\left(\tau,\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right)\right)$$

$$\times h_{x_i}\left(\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right)q\left(\hat{z}^m,\hat{\hat{z}}^m\right)\Big)d\tau$$

$$+\int\limits_{s\wedge\eta_n^m}^{s\wedge\eta^m}1_{(H)^c_{\epsilon_k}}(\tau,\hat{z}^m(\tau))\tilde{f}_r^i\left(\tau,\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right)h_{x_i}\left(\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right)q\left(\hat{z}^m,\hat{\hat{z}}^m\right)d\tau$$

$$+\int\limits_{t\wedge\eta^m}^{t\wedge\eta_n^m}1_{(H)^c_{\epsilon_k}}(\tau,\hat{z}^m(\tau))\tilde{f}_r^i\left(\tau,\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right)h_{x_i}\left(\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right)q\left(\hat{z}^m,\hat{\hat{z}}^m\right)d\tau\Bigg]+J$$

$$=\lim_{r\to\infty}\lim_{n\to\infty}(I_1+I_2+I_3+I_4+I_5+I_6)+J.$$

Let us estimate each term: $\lim_{r\to\infty}\lim_{n\to\infty}(|I_2|+|I_3|+|I_5|+|I_6|)=0$, since $s\wedge\eta_n^m\to_{n\to\infty}s\wedge\eta^m$ a.s., $t\wedge\eta_n^m\to_{n\to\infty}t\wedge\eta^m$ a.s.; by Lemma 2 and Corollary 2, we have

$$\lim_{r\to\infty}\lim_{n\to\infty}|I_4|\le C_2\lim_{r\to\infty}\Bigg(\int\limits_{([0,t]\times B_1(0,m))\cap(H)^c_{\epsilon_k/2}}\sup_{\|\hat{\hat{x}}\|\le m}\left(\det a\left(\tau,\hat{x},\hat{\hat{x}}\right)\right)^{-1}$$

$$\times\sup_{\|\hat{\hat{x}}\|\le m}\left|f^i\left(\tau,\hat{x},\hat{\hat{x}}\right)-\tilde{f}_r^i\left(\tau,\hat{x},\hat{\hat{x}}\right)\right|^{l+1}d\tau\,d\hat{x}\Bigg)^{1/(l+1)}=0.$$

Let us show that

$$\lim_{r\to\infty}\lim_{n\to\infty}|I_1|=0. \tag{18}$$

For each positive integer k, we construct a sequence of continuous functions $\varphi_j:R_+\times R^l\to[0,1]$ such that $\varphi_j\le 1_{[H]^c_{\epsilon_k}}$, $\varphi_j\uparrow_{j\to\infty}1_{(H)^c_{\epsilon_k}}$. By Corollaries 1 and 2,

$$\lim_{j\to\infty}\lim_{r\to\infty}\limsup_{n\to\infty}E\Bigg|\Bigg(\int\limits_{s\wedge\eta^m}^{t\wedge\eta^m}\left(1_{(H)^c_{\epsilon_k}}(\tau,\hat{z}_n^m(\tau))-\varphi_j(\tau,\hat{z}_n^m(\tau))\right)$$

$$\times\tilde{f}_r^i\left(\tau,\hat{z}_n^m(\tau),\hat{\hat{z}}_n^m(\tau)\right)h_{x_i}\left(\hat{z}_n^m(\tau),\hat{\hat{z}}_n^m(\tau)\right)d\tau\Bigg)q\left(\hat{z}_n^m,\hat{\hat{z}}_n^m\right)\Bigg|$$

$$\le C_3\lim_{j\to\infty}\limsup_{n\to\infty}E\left(\int\limits_{s\wedge\eta^m}^{t\wedge\eta^m}1_{(H)^c_{\epsilon_k}}(\tau,\hat{z}_n^m(\tau))\left(1_{(H)^c_{\epsilon_k}}(\tau,\hat{z}_n^m(\tau))-\varphi_j(\tau,\hat{z}_n^m(\tau))\right)d\tau\right)$$

$$\le C_4\lim_{j\to\infty}\Bigg(\int\limits_{([0,t]\times B_1(0,m))\cap(H)^c_{\epsilon_k}}\sup_{\|\hat{\hat{x}}\|\le m}\left(\left(\det a\left(\tau,\hat{x},\hat{\hat{x}}\right)\right)^{-1}\right.$$

$$\left.\times\left(1_{(H)^c_{\epsilon_k}}(\tau,\hat{x})-\varphi_j(\tau,\hat{x})\right)^{l+1}\right)d\tau\,d\hat{x}\Bigg)^{1/(l+1)}=0, \tag{19}$$

$$\lim_{j\to\infty}\lim_{r\to\infty} E\Bigg(\Bigg(\int\limits_{s\wedge\eta^m}^{t\wedge\eta^m}\Big(1_{(H)^c_{\epsilon_k}}(\tau,\hat{z}^m(\tau))-\varphi_j(\tau,\hat{z}^m(\tau))\Big)$$
$$\times \tilde{f}^i_r\left(\tau,\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right)h_{x_i}\left(\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right)d\tau\Bigg)q\left(\hat{z}^m,\hat{\hat{z}}^m\right)\Bigg)=0. \quad (20)$$

Since $\left(\hat{z}^m_n(\tau),\hat{\hat{z}}^m_n(\tau)\right)\to_{n\to\infty}\left(\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right)$ uniformly with respect to $\tau\in[0,t]$ with probability 1, we have

$$\lim_{j\to\infty}\lim_{r\to\infty}\lim_{n\to\infty} E\Bigg(\int\limits_{s\wedge\eta^m}^{t\wedge\eta^m}\Big(\varphi_j(\tau,\hat{z}^m_n(\tau))\tilde{f}^i_r\left(\tau,\hat{z}^m_n(\tau),\hat{\hat{z}}^m_n(\tau)\right)h_{x_i}\left(\hat{z}^m_n(\tau),\hat{\hat{z}}^m_n(\tau)\right)q\left(\hat{z}^m_n,\hat{\hat{z}}^m_n\right)$$
$$-\varphi_j(\tau,\hat{z}^m(\tau))\tilde{f}^i_r\left(\tau,\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right)h_{x_i}\left(\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right)q\left(\hat{z}^m,\hat{\hat{z}}^m\right)\Big)d\tau\Bigg)=0. \quad (21)$$

Relation (18) readily follows from (19)–(21). The proof of (16) and hence of (15) is complete.

There exists a sequence $k_n\to+\infty$, $n\to\infty$, such that

$$\lim_{n\to\infty}E\Bigg(\Bigg(\int\limits_{s\wedge\eta^m_n}^{t\wedge\eta^m_n}1_{(H)^c_{\epsilon_{k_n}}}(\tau,\hat{z}^m_n(\tau))f^i_n\left(\tau,\hat{z}^m_n(\tau),\hat{\hat{z}}^m_n(\tau)\right)h_{x_i}\left(\hat{z}^m_n(\tau),\hat{\hat{z}}^m_n(\tau)\right)d\tau\Bigg)q\left(\hat{z}^m_n,\hat{\hat{z}}^m_n\right)\Bigg)$$
$$=E\Bigg(\Bigg(\int\limits_{s\wedge\eta^m}^{t\wedge\eta^m}1_{H^c}(\tau,\hat{z}^m(\tau))f^i\left(\tau,\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right)h_{x_i}\left(\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right)d\tau\Bigg)q\left(\hat{z}^m,\hat{\hat{z}}^m\right)\Bigg). \quad (22)$$

From Lemma 2.5 in [5], relation (22), and the continuity of the function $f^i(t,X)$ on the set

$$\{(t,x_1,\ldots,x_d)\ |\ (t,x_1,\ldots,x_l)\in H(1,\ldots,l)\},$$

we obtain the relation

$$\lim_{n\to\infty}E\Bigg(\Bigg(\int\limits_{s\wedge\eta^m_n}^{t\wedge\eta^m_n}1_{(H)_{\epsilon_{k_n}}}(\tau,\hat{z}^m_n(\tau))f^i_n\left(\tau,\hat{z}^m_n(\tau),\hat{\hat{z}}^m_n(\tau)\right)h_{x_i}\left(\hat{z}^m_n(\tau),\hat{\hat{z}}^m_n(\tau)\right)d\tau\Bigg)q\left(\hat{z}^m_n,\hat{\hat{z}}^m_n\right)\Bigg)$$
$$=E\Bigg(\Bigg(\int\limits_{s\wedge\eta^m}^{t\wedge\eta^m}1_{H}(\tau,\hat{z}^m(\tau))f^i\left(\tau,\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right)h_{x_i}\left(\hat{z}^m(\tau),\hat{\hat{z}}^m(\tau)\right)d\tau\Bigg)q\left(\hat{z}^m,\hat{\hat{z}}^m\right)\Bigg). \quad (23)$$

By taking into account (22), (23), and the relation $P^{\Psi_n}=P^{\varepsilon_n}$, we get

$$\lim_{n\to\infty}E\Bigg(\Bigg(\int\limits_{s\wedge\tau^m_n}^{t\wedge\tau^m_n}f^i_n(\tau,X^m_n(\tau))h_{x_i}(X^m_n(\tau))d\tau\Bigg)q(X^m_n)\Bigg)$$
$$=E\Bigg(\Bigg(\int\limits_{s\wedge\eta^m}^{t\wedge\eta^m}f^i(\tau,z^m(\tau))h_{x_i}(z^m(\tau))d\tau\Bigg)q(z^m)\Bigg). \quad (24)$$

By using similar considerations for any fixed $i, j \in \{1, \ldots, d\}$, one can justify the relations

$$\lim_{n\to\infty} E\left(\left(\int\limits_{s\wedge\tau_n^m}^{t\wedge\tau_n^m} \sigma_n^{ij}\left(\tau, X_n^m(\tau)\right) h_{x_i x_j}\left(X_n^m(\tau)\right) d\tau\right) q\left(X_n^m\right)\right)$$
$$= E\left(\left(\int\limits_{s\wedge\eta^m}^{t\wedge\eta^m} \sigma^{ij}\left(\tau, z^m(\tau)\right) h_{x_i x_j}\left(z^m(\tau)\right) d\tau\right) q\left(z^m\right)\right). \tag{25}$$

From (14), (24), and (25), we obtain

$$E\left(\left(h\left(z^m(t)\right) - h\left(z^m(s)\right) - \int\limits_{s\wedge\eta^m}^{t\wedge\eta^m}\left(\frac{1}{2}\sum_{i,j=1}^{d} \sigma^{ij}\left(\tau, z^m(\tau)\right) h_{x_i x_j}\left(z^m(\tau)\right) + \sum_{i=1}^{d} f^i\left(\tau, z^m(\tau)\right) h_{x_i}\left(z^m(\tau)\right)\right) d\tau\right) q\left(z^m\right)\right) = 0;$$

therefore, the process

$$h(z(t)) - h(z(0)) - \int\limits_0^t \left(\frac{1}{2}\sum_{i,j=1}^{d} \sigma^{ij}(\tau, z(\tau)) h_{x_i x_j}(z(\tau)) + \sum_{i=1}^{d} f^i(\tau, z(\tau)) h_{x_i}(z(\tau))\right) d\tau$$

is a local $(\mathscr{F}_t)$-martingale.

As was shown in [6, pp. 159–160 of the Russian translation], on the extension $\left(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P}\right)$ with the flow $\tilde{\mathscr{F}}_t$ of the probability space $(\Omega, \mathscr{F}, P)$ with the flow $\mathscr{F}_t$, there exists an $\left(\tilde{\mathscr{F}}_t\right)$-Brownian motion $\tilde{W}(t)$ with $\tilde{W}(0) = 0$ a.s. such that the relation

$$z(t) = z(0) + \int\limits_0^t f(\tau, z(\tau)) d\tau + \int\limits_0^t g(\tau, z(\tau)) d\tilde{W}(\tau)$$

is valid with probability 1 for any $t \in [0, e)$. Consequently, $\left(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P}, \tilde{\mathscr{F}}_t, \tilde{W}(t), z(t), e\right)$ is a weak solution of Eq. (1). The proof of the theorem is complete.

Consider the following example:

$$dx_1(t) = \left(r\left(x_1(t)\right) + t x_2^2(t)\right) dt + r\left(x_2(t)\right) dW_1(t),$$
$$dx_2(t) = r\left(x_2(t) + 1\right) dt + x_2(t) dW_1(t),$$

where $r(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0. \end{cases}$ The function $\sigma = g g^{\mathrm{T}}$ is continuous; therefore, condition A is valid for it. Consider the function f. For the function $f^{(1)}\left(t, x_1, x_2\right) = r\left(x_1\right) + t x_2^2$, we choose the first row of the matrix g, $H(1)$ is an empty set, and for $f^{(2)}\left(t, x_1, x_2\right) = r\left(x_2 + 1\right)$, we choose the second row of the matrix g; obviously, the set

$$H(2) \times \{x_1 \in R\} = \{(t, x_1, x_2) \mid t \in R_+,\ x_1 \in R,\ x_2 = 0\}$$

is contained in the set of points of continuity of the mapping $f^{(2)}$. Consequently, the function f satisfies condition A. By the theorem in the present paper, for any given probability ν on $\left(R^d, \mathscr{B}\left(R^d\right)\right)$, there exists a weak solution with the initial distribution ν. Note that known theorems [1–5] do not imply the existence of weak solutions of the system in question.

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