Artinian serial rings

Gennadi Puninski
Moscow State Social University, Losinoostrovskaya 24,
107150 Moscow, Russia
E-mail: punins@orc.ru

1 Introduction

A ring is serial if it admits a left and a right decomposition into a direct sum of uniserial modules. Artinian serial rings were introduced by Asano and Köthe and were considered as "generalized uni–serial rings" by Nakayama [13]. A natural example is a ring $T_n(D)$ of upper triangular matrices over a (skew) field D. It is also often that a group ring $\mathbf{k}G$ of a finite group G over a field $\mathbf{k}$ is serial. Besides, every proper factor of a hereditary noetherian prime ring is of this kind by Eisenbud and Griffith [3].

Artinian serial rings also provides us by a natural example of rings of a finite representation type. Precisely by Nakayama [14] every module over such a ring is a direct sum of uniserial cyclic modules and there is only finite number of those up to isomorphism.

So the theory of modules over artinian serial rings seems to be completely clear. Slightly less is known on the structure of artinian serial rings itselves. Goldie [6] proved that a nonsingular artinian serial ring is a direct sum of blocked upper triangular matrix rings over skew fields. There is also some improvement of this result by Murase [10] : every indecomposable artinian serial ring with a simple projective module is a homomorphic image of a blocked upper triangular matrix ring over a skew field. Note that the homomorphic images of a ring $T_n(D)$ could be easily described using "star sets" (see [2]).

An essential progress was made also by using so called Kuppisch series (see Kuppisch [8]) i.e. a specially ordered sequence of lengths of indecomposable projective modules e_iR. On this way Murase [10] completely described artinian serial rings such that the length of every module e_iR does not exceed the Goldie dimension of R (rings of first kind in his terminology). There is also a result of Fuller [4] along these lines: every serial finite dimensional algebra over a perfect field is isomorphic to a semigroup algebra of a very easily described semigroup.

Note that Eisenbud and Griffith [2] splitted indecomposable artinian serial rings in four classes 1) artinian semisimple rings; 2) artinian principal ideal rings; 3) homomorphic images of blocked upper triangular matrices over a skew field and 4) which is essentially the complement of 1)–3).

In this paper we will present the theory of artinian serial rings in an unified way. In particular following to Kiev's school approach (see for instance Kirichenko [7]) we will permanently use the notion of a quiver similarly to the theory of finite dimensional algebras. We show that the quiver of an artinian serial ring R is a disjunct union of circles and lines, where only one component occurs if R is indecomposable. What follows is a description of the above splitting theorem in terms of the quiver. Precisely the four cases in this result correspond to a shape of the quiver: 1) a point; 2) a loop; 3) a line of length at least two; 4) a circle of length at least two.

The main theme for the further researches in this paper is how to find a hidden triangular matrix construction into an artinian serial ring. The main technical ingredient is so called "blow–up" construction which was made explicit in Kirichenko [7] and implicit in Müller [9]. On this way we show that every indecomposable artinian serial ring with a circle quiver is a homomorphic image of a ring R, where R is obtained from a quasi–Frobenius ring by finite many blow–ups. So the main difficulty is to describe serial QF-rings. Note that serial QF-rings can be characterized as artinian serial rings with a constant Kuppisch series. Also every serial group ring of a finite group over a field is QF.

We show that except one particular case a basic indecomposable serial QF-ring is a kind of semigroup ring over an artinian uniserial ring. Between exceptional examples of serial QF-rings we find one which is not a homomorphic image of any hereditary noetherian prime ring.

We also investigate and give a complete description of artinian serial rings with a faithful indecomposable module (we will call them d-rings). They also are characterized in terms of their Kuppisch series: this sequence is decreasing with a difference one. In particular, every artinian serial ring is a subdirect product of d-rings.

2 Basic facts

A module M is called *uniserial* if its lattice of submodules is a chain. M is said to be *serial* if M is a direct sum of uniserial modules. A ring R is *right (left) (uni-) serial* if the module R_R ($_RR$) is (uni-) serial. Finally R is (uni-) serial if it is left and right (uni-) serial. Thus R is a serial ring if there is a decomposition $1 = e_1 + \cdots + e_n$ into a sum of pairwise orthogonal idempotents such that all right modules e_iR are uniserial and all left modules Re_i are uniserial. In particular n is equal to the right and left Goldie dimension of R.

Since every serial ring is semiperfect, hence the collection $e_1, \ldots, e_n$ is defined up to conjugation by an unit of R (see [16, Thm. 2.9.18]). If R is a serial ring, R_{ij}, $i, j = 1, \ldots, n$ will denote an abelian group e_iRe_j. Then $R_i = R_{ii}$ is a ring and R_{ij} is an R_i–R_j-bymodule. The following is an easy criterion how to check seriality of a given semiperfect ring.

Fact 2.1 *[9, L. 1], [1, Cor.]A semiperfect ring $R = (R_{ij})$ is serial iff for every $r \in R_{ij}$, $s \in R_{ik}$ there are $u \in R_{kj}$, $v \in R_{jk}$ such that either $r = su$ or $s = rv$ holds.*

So the following example can be verified easily.

Example 2.2 *Every artinian semisimple ring is serial so as the ring $T_n(D)$ of upper triangular matrices over a (skew) field D.*

A semiperfect ring R with a decomposition $R = e_1R \oplus \ldots \oplus e_nR$ into a sum of local modules is called *basic* if $e_iR \ncong e_jR$ for $i \neq j$. Every semiperfect ring contains a basic subring S and R can be obtained from S by blocking as in the following example:

$$S = \begin{pmatrix} S_1 & S_{12} \\ S_{21} & S_2 \end{pmatrix} \Longrightarrow \begin{pmatrix} S_1 & S_{12} & S_{12} \\ S_{21} & S_2 & S_2 \\ S_{21} & S_2 & S_2 \end{pmatrix} = R\,.$$

The following claim is an essential part of the foregoing blow-up construction.

Lemma 2.3 *Let R be an artinian uniserial ring with the Jacobson radical* J. *Then*

$$S = \begin{pmatrix} R & R & \ldots & R \\ \mathrm{J} & R & \ldots & R \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{J} & \mathrm{J} & \ldots & R \end{pmatrix}$$

is a basic indecomposable artinian serial ring.

Proof. If e_i is the diagonal matrix unit e_{ii}, then all rings $S_i = e_i S e_i = R$ are local, hence R is semiperfect. The seriality follows easily by using Fact 2.1. Also by [5, L. 5.5] a serial ring is artinian iff all the diagonal rings are artinian. S is basic since $S_{ij} S_{ji} = \mathrm{J} \subset R = S_i$ for all $i \neq j$. $\square$

So we are going to introduce B. Müller's "blow–up" construction. Let $R = (R_{ij})$ be an (artinian) serial ring. Then S is a "blow–up" of R if it is obtained from R by 1) blocking some diagonal component R_i and then 2) replacing this block by triangular matrices over R_i with $\mathrm{J}_i = \mathrm{J}(R_i)$ everywhere downward the main diagonal. The following example illustrates this concept:

$$\begin{pmatrix} R_1 & R_{12} \\ R_{21} & R_2 \end{pmatrix} \Longrightarrow \begin{pmatrix} R_1 & R_1 & R_1 & R_{12} \\ \mathrm{J}_1 & R_1 & R_1 & R_{12} \\ \mathrm{J}_1 & \mathrm{J}_1 & R_1 & R_{12} \\ R_{21} & R_{21} & R_{21} & R_2 \end{pmatrix}$$

Similarly to Lemma 2.3 we obtain that S is an artinian serial ring which is basic if R does.

For a module M, $\mathrm{Gd}(M)$ will denote its Goldie dimension. Since for a serial ring R, $\mathrm{Gd}(R_R) = \mathrm{Gd}({}_R R)$ we will write $\mathrm{Gd}(R)$ instead. Let M be a right module over a ring R, $m \in M$. Put $\mathrm{ann}(m)(R) = \{r \in R \mid mr = 0\}$ which is a right ideal of R. To distinquish a side, we will write $\mathrm{ann}(S)(m)$ if M is a left S-module.

A ring R is called quasi–Frobenius (QF) if it is (left and right) artinian and (left and right) self–injective.

3 A quiver

Let $R = (R_{ij})$ be a basic artinian serial ring with a collection of indecomposable orthogonal idempotents $e_1, \ldots, e_n$. Since every module $e_i R$ is uniserial, its Jacobson radical $\mathrm{J}(e_i R)$ is principal. We put $i \rightsquigarrow j$ if $\mathrm{J}(e_i R) = rR$ for some $0 \neq r \in R_{ij}$; $i, j = 1, \ldots, n$. The obtained direct graph $\Gamma(R)$ is called *a quiver* of R. For an arbitrary serial ring R we define a quiver of R as the quiver of its basic subring. The following lemma shows that the shape of this graph is very special.

Lemma 3.1 *For every basic artinian serial ring R its quiver $\Gamma(R)$ is a disjunct union of circles and lines. Moreover, R is indecomposable iff $\Gamma(R)$ is connected, thus is either a circle or a line.*

Proof. For the former claim it suffices to prove that every point i is a source of at most one arrow and a sink of at most one arrow. By the way of contrary suppose that $i \rightsquigarrow j, k$ for $j \neq k$, hence $\mathrm{J}(e_iR) = rR = sR$ for $0 \neq r \in R_{ij}$, $0 \neq s \in R_{ik}$. Since e_iR is an uniserial module, we may assume that $s \in rR$, hence $s = rt$ for $t \in R_{jk}$. Since $j \neq k$ and R is basic, hence $t \in \mathrm{J}(R)$ which implies $r \in \mathrm{J}^2(R)$, a contradiction. Similarly $j, k \rightsquigarrow i$ yields $j = k$.

For the latter statement R is decomposable iff there is a partition of $\{1, \ldots, n\}$ in two parts such that $R_{ij} = R_{ji} = 0$ for i, j being in different parts. Since $i \rightsquigarrow j$ clearly implies $R_{ij} \neq 0$, hence $\Gamma(R)$ is connected yields that R is indecomposable.

Supposing that $\Gamma(R)$ is not connected let us choose i, j from distinct components of $\Gamma(R)$ and $0 \neq r \in R_{ij}$, $r \in \mathrm{J}^m(R)$, where one may assume that m is the least between elements with this property. In particular, $\mathrm{J}(e_iR) \neq 0$, hence $\mathrm{J}(e_iR) = sR$ for $0 \neq s \in R_{ik}$. Therefore $i \rightsquigarrow k$ yields $r = st$ for $t \in R_{kj}$ and $t \notin \mathrm{J}^m(R)$, a contradiction. □

Note that an artinian serial ring R has the same quiver as $R/\mathrm{J}^2(R)$. If $0 \neq r \in R_{ij}$ is such that $rR = \mathrm{J}(e_iR)$, we will say that r defines the arrow $i \rightsquigarrow j$. Then clearly $Rr = \mathrm{J}(Re_j)$, hence r defines the arrow $j \rightsquigarrow i$ in the left quiver of R.

Also i is a source for no arrow in $\Gamma(R)$ iff e_iR is a simple (projective) module. For instance if the quiver of R is a point, then all modules e_iR are simple and isomorphic, hence R is an artinian simple ring, which corresponds to the case 1) in the above classification by Eisenbud and Griffith. If the quiver of R is a loop, a basic subring of R is uniserial and not a skew field, hence R is a full matrix ring over an uniserial ring (case 2) in the above classification). The rest shapes of the quiver are 3) a line with at least two points and 4) a circle with at least to points. To analyze them the following general construction will be useful.

Lemma 3.2 *Let M be an S–R-bymodule and $m \in M$ such that $mR = Sm$. Then $\mathrm{ann}(S)(m)$, $\mathrm{ann}(m)(R)$ are two–sided ideals and there is a natural isomorphism of rings $S/\mathrm{ann}(S)(m) \cong R/\mathrm{ann}(m)(R)$.*

Proof. $\mathrm{ann}(m)(R)$ is cleary a right ideal in R. Suppose that $mr = 0$ for some $r \in R$ and $u \in R$. Then $mu = sm$ for some $s \in S$, hence $mur = smr = 0$ and $ur \in R$. Similarly $\mathrm{ann}(S)(m)$ is an ideal in S. Then the rule $mr = sm$ clearly defines the required isomorphism. □

The modelling situation where the Lemma 3.2 will be applied is the following. Let $R = (R_{ij})$ be a basic artinian serial ring and

$$S = \begin{pmatrix} R_1 & \dots & R_{1,n-2} & R_{1,n-1} & R_{1,n-1} \\ R_{21} & \dots & R_{2,n-2} & R_{2,n-1} & R_{2,n-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ R_{n-1,1} & \dots & R_{n-1,n-2} & R_{n-1} & R_{n-1} \\ R_{n-1,1} & \dots & R_{n-1,n-2} & \mathrm{J}(R_{n-1}) & R_{n-1} \end{pmatrix}$$

which is a blow–up of a (basic artinian serial) ring $R|_{n-1} = (R_{ij})_{1\le i,j\le n-1}$. Suppose that $z \in R_{n-1,n}$ defines the arrow $n-1 \rightsquigarrow n$. Let

$$Z = \begin{pmatrix} 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \\ 0 & \dots & 0 & z \end{pmatrix}$$

be an element of S–R-bymodule

$$\begin{pmatrix} R_1 & \dots & R_{1,n-1} & R_{1,n} \\ \vdots & \ddots & \vdots & \vdots \\ R_{n-1,1} & \dots & R_{n-1} & R_{n-1,n} \\ R_{n-1,1} & \dots & R_{n-1} & R_{n-1,n} \end{pmatrix}$$

Then we obtain

$$ZR = \begin{pmatrix} R_1 & \dots & R_{1,n-1} & R_{1n} \\ \vdots & \ddots & \vdots & \vdots \\ R_{n-1,1} & \dots & R_{n-1} & R_{n-1,n} \\ zR_{n1} & \dots & zR_{n,n-1} & zR_n \end{pmatrix} =$$

$$= \begin{pmatrix} R_1 & \dots & R_{1,n-1} & R_{1n} \\ \vdots & \ddots & \vdots & \vdots \\ R_{n-1,1} & \dots & R_{n-1} & R_{n-1,n} \\ R_{n-1,1} & \dots & \mathrm{J}(R_{n-1}) & R_{n-1,n} \end{pmatrix} =$$

$$= \begin{pmatrix} R_1 & \dots & R_{1,n-1} & R_{1,n-1}z \\ \vdots & \ddots & \vdots & \vdots \\ R_{n-1,1} & \dots & R_{n-1} & R_{n-1}z \\ R_{n-1,1} & \dots & \mathrm{J}(R_{n-1}) & R_{n-1}z \end{pmatrix} = SZ$$

Indeed since $R_{n-1,i} \subseteq \mathrm{J}(e_{n-1}R) = zR$ for $i \neq n-1$, hence $zR_{ni} = R_{n-1,i}$. Similarly $zR_{n,n-1} = \mathrm{J}(R_{n-1})$. Note that $\mathrm{ann}(Z)(R) = \mathrm{ann}(z)(e_nR)$ and $\mathrm{ann}(S)(Z) = \mathrm{ann}(R|_{n-1}e_{n-1})(z)$. Thus by Lemma 3.2 we obtain an isomorphism

$$S/\mathrm{ann}(R|_{n-1}e_{n-1})(z) \cong R/\mathrm{ann}(z)(e_nR)\,. \tag{1}$$

4 A line quiver

Let R be an indecomposable artinian serial ring whose quiver is a line with at least two points. We may assume that R is basic and arrange the idempotents such that $1 \rightsquigarrow 2 \rightsquigarrow \ldots \rightsquigarrow n$, hence the unique simple projective R-module is e_nR.

Theorem 4.1 *(see Murase [10] or Fuller [4, Thm. 32.7])Every basic indecomposable artinian serial ring R with a line quiver and of Goldie dimension n is a homomorphic image of a ring $T_n(D)$ over a skew field D.*

Proof. Let $r_i \in R_{i,i+1}$ for $i = 1, \ldots, n-1$ define the arrow $i \rightsquigarrow i+1$ and $z = r_{n-1}$. By the construction of the previous section $R/\mathrm{ann}(z)(e_nR) \cong S/\mathrm{ann}(R|_{n-1}e_{n-1})(z)$, where S is a blow–up of $R|_{n-1}$. We prove that $\mathrm{ann}(z)(e_nR) = 0$. Every element of R_{ni} defines by left multiplication a homomorphism $e_iR \to e_nR$. Since e_nR is a simple projective module, hence $R_{ni} = 0$ for $i < n$ and R_n is a skew field, which yields the desired.

Thus R is a homomorphic image of S. Additionally $R|_{n-1}$ is clearly a basic indecomposable artinian serial ring with the quiver $1 \rightsquigarrow \ldots \rightsquigarrow n-1$, where the arrow $i \to i+1$ is defined by r_i, $i < n-1$. Then as above R_{n-1} is a skew field, hence the result follows by induction. □

Note that the homomorphic images of $T_n(D)$ are easily described (see [2]). What follows is the complete description of indecomposable artinian serial rings with a line quiver.

Corollary 4.2 *Every indecomposable artinian serial ring with a line quiver is a homomorphic image of a blocked upper triangular matrix ring over a skew field.*

Thus we are able to verify the following theorem by Goldie.

Theorem 4.3 *([6, Thm. 8.11])Every indecomposable nonsingular artinian serial ring is isomorphic to a blocked upper triangular matrix ring over a skew field.*

Proof. We may assume that R is basic and prove that $R \cong T_n(D)$. Clearly R has a line quiver. Indeed otherwise let $r_i \in R_{i,i+1}$ define the arrow $i \rightsquigarrow i+1$, $i < n$ and r_n defines the arrow $n \rightsquigarrow 1$. Myltiplying along the circle $r_1 \cdot r_2$, $r_1 \cdot r_2 \cdot r_3$, $\ldots$, we find $0 \neq r \in R_{ij}$, $0 \neq s \in R_{jk}$ such that $rs = 0$, which is impossible since R is nonsingular.

Following to the proof of Theorem 4.1 by nonsingularity we obtain $\mathrm{ann}(R|_{n-1}e_{n-1})(z) = 0$ in (1), hence $R \cong S$ which yields the desired. $\square$

5 A circle quiver

Let R be a basic serial ring with a circle quiver. The idempotents of R can be arranged such that $1 \rightsquigarrow 2 \rightsquigarrow \ldots \rightsquigarrow n \rightsquigarrow 1$. Let us define the (right) Kuppisch series of R as the sequence $(c_1, \ldots, c_n)$, where c_i is the length of e_iR. Let $r_i \in R_{i,i+1}$, $i < n$ define the arrow $i \rightsquigarrow i+1$ and r_n defines the arrow $n \rightsquigarrow 1$. Clearly c_i is the largest k such that the product $r_i \cdot \ldots \cdot r_{i+k-2}$ is nonzero. What immediately follows is

Remark 5.1 *Let R be a basic artinian serial ring with a circle quiver and the right Kuppisch series $(c_1, \ldots, c_n)$. Then 1) $c_i \geq 2$ for every i ; 2) $c_{i+1} \geq c_i - 1$ for $i < n$ and $c_1 \geq c_n - 1$.*

It follows that $|c_i - c_j| < n$ for all i, j and this series is either constant or $c_{k+1} = c_k - 1$ for some k.

Note that the Kuppisch series can be defined similarly also for artinian serial ring with a line quiver. For instance for a ring $T_n(D)$ its right Kuppisch series is $(n, n-1, \ldots, 1)$ and the left Kuppisch series is $(1, 2, \ldots, n)$. Also let $\mathbb{Z}_4$ be the factor of integers $\mathbb{Z}$ by the ideal $4\mathbb{Z}$. Then the ring

$$R = \begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ 2\mathbb{Z}_4 & \mathbb{Z}_4 \end{pmatrix}$$

is basic artinian serial with a circle quiver $1 \rightsquigarrow 2 \rightsquigarrow 1$ and the right Kuppisch series $(4, 3)$. Indeed R is finite, hence artinian and R is serial by Lemma 2.3. Clearly $\mathrm{J}(e_1R) = r_1R$ where $r_1 = e_{12}$ and $\mathrm{J}(e_2R) = r_2R$ for $r_2 = 2 \cdot e_{21}$. Thus $r_1 \cdot r_2 \cdot r_1 = 2e_{12} \neq 0$ and $r_1r_2r_1r_2 = 0$, hence the length of e_1R is 4. Similarly $r_2r_1 = 2e_2 \neq 0$ and $r_2r_1r_2 = 0$.

The following fact characterizes serial QF-rings in terms of their Kuppisch series.

Lemma 5.2 *([10, Thm. 1.9])A basic artinian serial ring with a circle quiver is* QF *iff its (right) Kuppisch series is constant.*

Our next aim is to lift serial rings under consideration to serial QF-rings.

Theorem 5.3 *Let R be a basic artinian serial ring with a circle quiver. Then R is a homomorphic image of a ring S of the same Goldie dimension which is obtained from a basic indecomposable serial* QF*-ring by finite many blow–ups.*

Proof. By Remark 5.1 one may assume that $c_n = c_{n-1} - 1$ where $n = \mathrm{Gd}(R)$. By the construction of Section 3, we obtain the isomorphism (1). Check that $\mathrm{ann}(z)(e_n R) = 0$. Indeed otherwise $r_{n-1}s = 0$ for some $0 \neq s \in e_n R$. Decomposing s along the circle we may take $s = r_n \cdot r_1 \cdot \ldots \cdot r_i$, where the index i is defined modulo n if this product is too long. Since $s \neq 0$, hence $i + 1 < c_n$. But because $r_{n-1}s = 0$, hence $i + 2 \geq c_{n-1}$ which yields $c_n > i + 1 \geq c_{n-1} - 1 = c_n$, a contradiction.

Thus R is a homomorphic image of a ring S, $\mathrm{Gd}(S) = n$, which is a blow–up of $R|_{n-1}$. Clearly $R|_{n-1}$ is a basic artinian serial ring whose quiver $1 \rightsquigarrow \ldots \rightsquigarrow n-2 \rightsquigarrow n-1 \rightsquigarrow 1$ is defined by the elements $r_1, \ldots, r_{n-2}, r_{n-1} \cdot r_n$. Now the proof can be finished by an easy induction. □

Note that the homomorphic images of a given artinian serial ring can be easily described in terms of its Kuppisch series. Thus the structure of a basic artinian serial ring with a circle quiver can be determined by the structure of its QF "cover". Let us introduce the useful construction (cp. Fuller [4]).

For $m, n \geq 2$ let us define a semigroup $G = G(m, n)$ with zero 0. The elements of G are $0, e_1, \ldots, e_n$ and the words $w = r_i \cdot r_{i+1} \cdot \ldots \cdot r_j$ of length $\leq m - 1$ where the indices are defined modulo n. The multiplication table is given by 1) $e_k \cdot e_l = 0$ for $k \neq l$ and $e_k \cdot e_k = e_k$; 2) $e_k \cdot w = w$ if $k = i$ and 0 otherwise ; 3) $we_l = w$ if $l = j$ and 0 otherwise ; 4) if $v = r_k \cdot r_{k+1} \cdot \ldots \cdot r_l$ then $w \cdot v = wv$ if $j = k$ and the length of wv is less then m, and $wv = 0$ otherwise. It is not difficult to prove that G is an (associative) semigroup.

Let us write $m = kn + l$, $l < n$ and let V be an artinian uniserial ring with the Jacobson radical J, $\mathrm{J}^k \neq 0$, $\mathrm{J}^{k+1} = 0$ and $\mathrm{J} = pV = Vp$ for $p \in V$. We consider the semigroup ring VG, where $vg = gv$ for every $v \in V$, $g \in G$ with the additional relations: if $k \geq 1$ then for every $i = 1, \ldots, n$ put $r_i \cdot r_{i+1} \cdot \ldots \cdot r_{i-1} = p$ (the length of this word is n). Thus every element of the obtained ring S can be uniquely written as $r \cdot w$, where $r \in V \setminus \mathrm{J}$ and

$w = r_i \cdot \ldots \cdot r_j$ has a length at most $m-1$, and similarly in the form $v \cdot t$ for $t \in V \setminus \mathrm{J}$. It follows that S is an artinian serial ring with the right Kuppisch series $(m, \ldots, m)$, hence is a QF-ring. For instance if $k = 0$, then V is a (skew) field D and we get that S is the usual semigroup ring DG.

The following theorem describes "almost all" serial QF-rings as semigroup rings.

Proposition 5.4 *Let R be a basic indecomposable* QF *serial ring of Goldie dimension n with the constant Kuppisch series $(m, \ldots, m)$ where $m = kn+l$, $1 \neq l < n$. Then R is isomorphic to the just described ring S.*

Proof. Let r_i, $i = 1, \ldots, n-1$ define the arrow $i \rightsquigarrow i+1$ and r_n defines the arrow $n \rightsquigarrow 1$. The equality $R_i r_i = r_i R_{i,i+1}$ yields the isomorphism of rings $R_i/\mathrm{ann}(R_i)(r_i) = R_{i+1}/\mathrm{ann}(r_i)(R_{i+1})$. We prove that both annihilators are zero. Indeed otherwise by symmetry $r_i s = 0$ for some $0 \neq s \in R_{i+1}$ where we may assume that $s = r_{i+1} \cdot \ldots \cdot r_i$. Since $s \neq 0$, hence the length of the last word is less then m, hence is equal to $m-1$, since $r_i s = 0$. Thus n divides $m-1$, a contradiction.

Now all the rings R_i can be identified as in [4, p. 62]. Thus R has the desired structure. $\square$

It follows that if $l \neq 1$, an indecomposable basic serial QF-ring (so as its indecomposable factors) is uniquely determined by three datas: 1) the Goldie dimension n; 2) the right Kuppisch series $(m, \ldots, m)$, $m = kn + l$, $l < n$ and 3) the diagonal ring V, which is an artinian uniserial ring with the Jacobson radical J such that $\mathrm{J}^{k+1} = 0$. Moreover all the diagonal rings of R are isomorphic. The following example shows that is not the case if $m \equiv 1 \pmod{n}$.

Example 5.5 *Let R be a ring*

$$\begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2[x]/x^2 \end{pmatrix}$$

where the R_1-R_2 bymodule structure on the abelian groups $R_{12} = \mathbb{Z}_2$ is given by $2 \cdot e_{12} = e_{12} \cdot x = 0$, similarly for $R_{21} = \mathbb{Z}_2$, and $e_{12} \cdot e_{21} = 2$, $e_{21} \cdot e_{12} = x$. Then R is a basic serial QF*-ring with a quiver $1 \rightsquigarrow 2 \rightsquigarrow 1$ and the Kuppisch series $(3,3)$. Moreover R is not a factor of any hereditary noetherian prime ring.*

Proof. It can be easily checked using Fact 2.1 that R is an artinian serial ring. Clearly e_{12} defines the arrow $1 \rightsquigarrow 2$ and e_{21} realizes the arrow $2 \rightsquigarrow 1$.

Since $e_{12} \cdot e_{21} = 2 \neq 0$ and $e_{12} \cdot e_{21} \cdot e_{12} = 2 \cdot e_{12} = 0$, hence the length of $e_1 R$ is 3 and similarly the length of $e_2 R$ is 3. Also the diagonal rings $\mathbb{Z}_4$ and $\mathbb{Z}_2[x]/x^2$ are not isomorphic.

Suppose that R is a homomorphic image of a hereditary noetherian prime ring S. We may assume that S is indecomposable, hence by [15] it is obtained by blocking from the blow-ups of an artinian uniserial ring V. All the diagonal rings R_i are isomorphic to the factors of V. Since R_1 and R_2 have the same length, hence they should be isomorphic, a contradiction. □

6 d-rings

Let R be a basic indecomposable artinian serial ring of Goldie dimension n. We say that R is a d-ring if it has the right Kuppisch series $(m, m-1, \ldots, m-n+1)$, $m \geq n$. If $m = n$ then the right Kuppisch series of R is $(n, n-1, \ldots, 1)$, hence $R \cong T_n(D)$ by Section 4. Otherwise R has a circle quiver $1 \rightsquigarrow 2 \rightsquigarrow \ldots \rightsquigarrow n \rightsquigarrow 1$. For instance we have seen an example of a d-ring with the right Kuppisch series $(4, 3)$. Note also that Murase [12] considered so called "quasi–matrix" rings over a (skew) field $\mathbf{k}$. A typical example is given by a ring

$$S = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ 0 & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ 0 & 0 & \alpha_{11} & \alpha_{12} \\ 0 & 0 & 0 & \alpha_{22} \end{pmatrix}, \quad \alpha_i \in \mathbf{k} \tag{2}$$

It can be easily calculated that S is an artinian serial ring of Goldie dimension 2 (e_{11} and e_{22} are basic idempotents) with the right Kuppisch series $(4, 3)$. Hence S is a d-ring. Similarly every Murase's quasi–matrix ring is a homomorphic image of a d-ring.

Firstly we consider the case of Goldie dimension 2. Thus the right Kuppisch series of R is $(m, m-1)$ and its structure essentially depends on whether m is odd or even.

Lemma 6.1 *Let R be a basic artinian serial ring with a circle quiver and the right Kuppisch series $(2k, 2k-1)$, $k \geq 2$. Then the left Kuppisch series of R is $(2k-1, 2k)$ and R is isomorphic to the ring*

$$\begin{pmatrix} V & V \\ \mathrm{J} & V \end{pmatrix},$$

where V is an uniserial ring with the Jacobson radical J *such that* $\mathrm{J}^{k-1} \neq 0$ *and* $\mathrm{J}^k = 0$.

Proof. Let $r_1 \in R_{12}$ define the arrow $1 \rightsquigarrow 2$ and $r_2 \in R_{21}$ defines the arrow $2 \rightsquigarrow 1$. We have $(r_1 r_2)^{k-1} r_1 \neq 0$ and $(r_2 r_1)^{k-1} r_2 = 0$. It can be easily calculated that the left Kuppisch series of R is $(2k-1, 2k)$. Let us repeat the construction of Section 3 getting that $S/\mathrm{ann}(R_1)(r_1) \cong R/\mathrm{ann}(r_1)(e_2 R)$, where S is the blow–up of R_1. Since $c_2 = c_1 - 1$ similarly to the proof of Theorem 5.3 we obtain $\mathrm{ann}(r_1)(e_2 R) = 0$. Also $\mathrm{ann}(R_1)(r_1) = 0$ by symmetry. The Jacobson radical of R_1 is generated by $r = r_1 \cdot r_2$ and $r^{k-1} \neq 0$, $r^k = 0$ which yields the desired. □

Lemma 6.2 *Let R be a basic artinian serial ring with a circle quiver and the right Kuppisch series $(2k+1, 2k)$, $k \geq 1$. Then the left Kuppisch series of R is $(2k+1, 2k)$ and R is isomorphic to the ring*

$$\begin{pmatrix} V & V \\ \mathrm{J} & V \end{pmatrix}, \quad \textit{factorized by} \quad \begin{pmatrix} 0 & \mathrm{J}^k \\ 0 & \mathrm{J}^k \end{pmatrix},$$

where V is an uniserial ring with the Jacobson radical J *such that* $\mathrm{J}^k \neq 0$ *and* $\mathrm{J}^{k+1} = 0$.

Proof. Let $r_1 \in R_{12}$ define the arrow $1 \rightsquigarrow 2$ and $r_2 \in R_{21}$ defines the arrow $2 \rightsquigarrow 1$. Thus $(r_1 r_2)^k \neq 0$ and $(r_2 r_1)^k = 0$. Repeating the proof of preceding Lemma we get $\mathrm{ann}(r_1)(e_2 R) = 0$. But now $\mathrm{ann}(R_1)(r_1) = R_1 (r_1 r_2)^k = \mathrm{J}(R_1)^k$ which yields the desired isomorphism. □

Now let us consider the general case.

Theorem 6.3 *Let R be a basic artinian serial ring with the circle quiver and the right Kuppisch series $(m, m-1, \ldots, m-n+1)$, $m - n \geq 1$. Then either R is a blow–up of an uniserial ring (see Lemma 2.3) or it is isomorphic to the ring*

$$\begin{pmatrix} V & \ldots & V & V/\mathrm{J}^k & \ldots & V/\mathrm{J}^k \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathrm{J} & \ldots & V & V/\mathrm{J}^k & \ldots & V/\mathrm{J}^k \\ \mathrm{J} & \ldots & \mathrm{J} & V/\mathrm{J}^k & \ldots & V/\mathrm{J}^k \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathrm{J} & \ldots & \mathrm{J} & \mathrm{J}/\mathrm{J}^k & \ldots & V/\mathrm{J}^k \end{pmatrix},$$

where V is an uniserial ring with the Jacobson radical J *such that* $\mathrm{J}^k \neq 0$ *and* $\mathrm{J}^{k+1} = 0$.

Proof. Let r_i defines the arrow $i \rightsquigarrow i+1$, $i < n$ and r_n defines the arrow $n \rightsquigarrow 1$. Then $r_1 \cdot \ldots \cdot r_{m-1} \neq 0$ and $r_n \cdot r_1 \cdot \ldots \cdot r_i = 0$ where the length of this word is $m - n + 1$. Let us write the right Kuppisch series for R as $(m, \ldots, kn-1 \mid kn, \ldots, m-n+1)$. The direct calculation shows that the left Kuppisch series for R is $(kn-1, \ldots, m \mid m-n+1, \ldots, kn)$, i.e. it is obtained by the inversion of two pieces of the right Kuppisch series. For instance if the right Kuppisch series of R is $(12, 11 \mid 10, 9, 8)$ then its left Kuppisch series is $(11, 12 \mid 8, 9, 10)$.

Assume that the right hand part of the right Kuppisch series of R contains at least two members. Then as above we can apply the construction of Section 3 getting the isomorphism (1). Since $c_n = c_{n-1} - 1$ and $d_{n-1} = d_n - 1$, both annihilators in (1) are zero, hence $R \cong S$. Thus R is a blow up of $R|_{n-1}$. Moreover the right Kuppisch series for $R|_{n-1}$ is $(m-k, \ldots, k(n-1)-1 \mid k(n-1), \ldots, m-n-k)$ hence we can apply induction on the Goldie dimension.

Similarly if the left hand part of the right Kuppisch series of R has at least two members, we can find a blow–up structure using the element r_1 instead of r_n. Otherwise $\mathrm{Gd}(R) = 2$, hence we can use Lemmas 6.2 and 6.1. □

For instance if we are working at the concrete numerical example, we obtain the chain of reductions:

$$(12, 11 \mid 10, 9, 8) \to (10, 9 \mid 8, 7) \to (8, 7 \mid 6) \to (5 \mid 4)\,.$$

By Lemma 6.2 the last ring has the form

$$\begin{pmatrix} V & V \\ \mathrm{J} & V \end{pmatrix} \quad \text{factorized by} \quad \begin{pmatrix} 0 & \mathrm{J}^2 \\ 0 & \mathrm{J}^2 \end{pmatrix},$$

where V is an uniserial ring with the Jacobson radical J and $\mathrm{J}^2 \neq 0$, $\mathrm{J}^3 = 0$. Then R is isomorphic to the ring

$$\begin{pmatrix} V & V & V & V & V \\ \mathrm{J} & V & V & V & V \\ \mathrm{J} & \mathrm{J} & V & V & V \\ \mathrm{J} & \mathrm{J} & \mathrm{J} & V & V \\ \mathrm{J} & \mathrm{J} & \mathrm{J} & \mathrm{J} & V \end{pmatrix} \quad \text{factorized by} \quad \begin{pmatrix} 0 & 0 & \mathrm{J}^2 & \mathrm{J}^2 & \mathrm{J}^2 \\ 0 & 0 & \mathrm{J}^2 & \mathrm{J}^2 & \mathrm{J}^2 \\ 0 & 0 & \mathrm{J}^2 & \mathrm{J}^2 & \mathrm{J}^2 \\ 0 & 0 & \mathrm{J}^2 & \mathrm{J}^2 & \mathrm{J}^2 \\ 0 & 0 & \mathrm{J}^2 & \mathrm{J}^2 & \mathrm{J}^2 \end{pmatrix}$$

The following lemma shows that the class of d-rings is appeared very naturally.

Lemma 6.4 *A basic indecomposable artinian serial ring R is a d-ring iff R posseses of a faithful indecomposable module.*

Proof. Let R be a d-ring with the right Kuppisch series $(m, \dots, m-n+1)$. Then is can be easily shown that e_1R is a faithfull module. For the converse by Drozd–Warfield's theorem [1, Thm.], [17, Thm. 3.3] every indecomposable right module over R is a homomorphic image of a module e_iR. Thus we may assume that e_1R is faithful. Now if $c_i > c_1 - i + 1$ then the element $r_i \cdot \ldots \cdot r_j$ of length $c_i - 1$ annihilates e_1R, a contradiction. $\Box$

It follows by the above numerical example that there is a right d-ring which is not a left d-ring. The following Theorem describes left and right d-rings.

Theorem 6.5 *Let R be a basic indecomposable artinian serial ring. Then R is blow–up of an uniserial ring (see Lemma 2.3) iff the right Kuppisch series for R is $(kn, kn-1, \dots, kn-n+1)$, $k \geq 1$.*

Proof. If R is a blow–up of an uniserial ring S, then its right Kuppisch series is clearly of desired form. For the converse we repeat the proof of Theorem 6.3, reducing the situation to the case $n = 2$. Then the obtained right Kupisch series is $(2k, 2k-1)$, hence the Lemma 6.1 is applied. $\Box$

It follows from this theorem that every d-ring is uniquely determined by its Goldie dimension and the diagonal uniserial ring V. Also every basic indecomposable serial QF-ring as described in the Proposition 5.4 is a homomorphic image of a d-ring.

The following claim is a kind of structure theorem for artinian serial rings.

Corollary 6.6 *Every artinian serial ring R is a subdirect product of blocked d-rings.*

Proof. We may suppose that R is indecomposable and basic. Let I_i be the annihilator of e_iR and $S_i = R/I_i$. Then $\cap_i I_i = 0$, hence R is a subdirect product of the rings S_i. Moreover e_iR is a faithful indecomposable S_i-module, so the Lemma 6.4 can be applied. $\Box$

Note that every matrix ring over an artinian uniserial ring is serial but the converse is obviously not true. Nevertheless the preceding corollary and Theorem 6.3 yield

Corollary 6.7 *If R is an artinian serial PI-ring (i.e. a ring with a polynomial identity), then R lies in a variety generated by full matrix rings $M_{n_i}(R_i)$ over artinian uniserial PI-rings R_i.*

Note that even some "bad" artinian serial QF-rings are often factors of d-rings. For instance let k be a field of characteristic 3 and S_3 be the symmetric group. Then by [12] the group ring $\mathrm{k}S_3$ is isomorphic to a quasi–matrix ring (2) factorized by

$$\begin{pmatrix} 0 & 0 & 0 & \mathrm{k} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Applying Theorem 6.3, we get that $\mathrm{k}S_3$ is isomorphic to the ring

$$\begin{pmatrix} V & V \\ \mathrm{J} & V \end{pmatrix} \quad \text{factorized by} \quad \begin{pmatrix} 0 & \mathrm{J} \\ 0 & 0 \end{pmatrix},$$

where V is an uniserial ring with the Jacobson radical $\mathrm{J} \neq 0$ and $\mathrm{J}^2 = 0$.

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