5
Analytical Methods for Heat Conduction in Composites and Porous Media

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5.1
Introduction

The goal of this chapter is to describe analytical methods applied to the study of steady heat conduction in various types of composites and porous media. We present several analytical formulas for the effective (macroscopic) conductivity tensor which are deduced by using different approaches based on the recent results in the theory of partial differential equations and complex analysis. The study of effective characteristics has recently become a separate subject with its own philosophy and machinery. Composites and porous media differ by geometry and by the type of physical problems that appear. For composites, the most popular are problems of conductivity, elasticity, elastoplasticity and thermoelasticity (e.g. Refs. [1–4]), but for porous media, problems of fluid mechanics are mostly studied (e.g. Refs. [5–9]).

The analytical approach to the study of heat conduction allows us to unify partly the theory of the effective thermal properties in composite materials and porous media. In the present chapter, pure steady conductivity problems are considered when the filler of pores (fluid or gas) is static. Such problems are benchmarks of heat and mass transfer problems of the mechanics of porous media [10].

The main attention throughout this chapter will be paid to *analytic* or *constructive*, or *closed form solutions* to the above mentioned problems. Different interpretations can be given to such a notion. For us to get an analytical solution means to find the formula which contains a finite set of elementary and special functions, compositions, integrals, derivatives and even series. Besides, all objects in such a formula have to have a precise meaning (for instance, the type of the convergence of integrals and series should be described). Last, the domains of parameters, as well as all functions, integrals, etc., have to be explicitly determined. It will also be shown also that they (or their intersections, if necessary) are nonempty. This approach is slightly nontraditional. In classic books, it is supposed that series do not form closed form solutions, but special functions do. It leads to certain misunderstandings since not

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all special functions have integral representations. We suppose that our meaning of a closed form solution could give a way to work efficiently with the mathematical object involved in the solution formulas.

The above meaning of analytic solutions is very close to the sense of solutions obtained by certain numerical methods. To avoid misunderstandings we will distinguish between (pure) analytic solutions and those obtained by using certain analytic numerical procedure. The latter will be called *approximate analytic solutions*.

Among the numerical methods which can be called approximate analytical methods we have to point out the collocation method and its modifications as developed and applied to the study of composites by Kolodziej and co-workers [11–13], the finite element method presented in other chapters of this book, the integral equation method, in the form developed, for example, by Lifanov [14,15], as well as the successive approximation method and methods of decomposition, in particular, Schwarz's alternating method (all applied in the study of composites in Ref. [16]). Despite the method of truncation for infinite linear algebraic systems can be effective in numerical computations, one can hardly accept that this method yields a "closed form solution" as is frequently declared.

In this review, the main attention is paid to analytically exact and approximate formulas for the effective conductivity tensor. Other important questions as bounds [17], homogenization [1,2,17–20], coupled heat and mass transfer [5,6,10] can be found in the cited works.

5.2 Mathematical Models for Heat Conduction

5.2.1 General

Consider the Euclidian space $\mathbb{R}^M$ as a space of the spatial variable $\mathbf{x} = (x_1, x_2, \ldots, x_M)$. Usually $M = 3$. Sometimes due to the symmetry of the problem under discussion it is convenient to take $M = 2$ or $M = 1$, making corresponding changes in the equations. Let Ω be a domain occupied by the conducting material (composite or porous media). One of the most important objects in the mathematical theory of steady heat conduction is *the temperature distribution* $T(\mathbf{x})$ and *the heat flux* $\mathbf{q}(\mathbf{x})$. In physics, temperature is the measure of the energy possessed by particles (molecules, electrons, etc.) per unit volume of the material. The heat flux is the heat transfer rate (in unit of time) per unit volume. Below, the dimensions of the basic variables can be taken in SI units. The unit for temperature is the kelvin *K*, for heat flux it is $\mathrm{Jm}^{-2}\mathrm{s}^{-1}$, for conductivity it is $\mathrm{Wm}^{-1}\mathrm{K}^{-1}$, where *J* is the joule, *m* is the meter, *s* is the second, *W* is the watt.

From a mathematical point of view $T(\mathbf{x})$ is a scalar field depending on the variable $\mathrm{x} \in \mathbb{R}^M$, $\mathbf{q}(\mathbf{x}) = (q_1(\mathbf{x}), q_2(\mathbf{x}), q_3(\mathbf{x}))$ is a vector field. The equations representing dependence of the flux $\mathbf{q}(\mathbf{x})$ on the temperature $T(\mathbf{x})$ are called (the heat transfer) *constitutive* *relations*. In the linear case the constitutive relation for conducting material has the form of Fourier's law (e.g. Ref. [21])

$$\mathbf{q} = -\Lambda \nabla T \tag{5.1}$$

where ∇T is the gradient of $T(\mathbf{x})$ and Λ is a tensor. In Cartesian coordinates $\nabla T = \left(\frac{\partial T}{\partial x_1}, \frac{\partial T}{\partial x_2}, \frac{\partial T}{\partial x_3},\right)$.

The constitutive relation (5.1) means a (local) proportionality of the flux and the gradient of temperature distribution. In the linear case, the proportionality coefficient Λ depends solely on the spatial variable $\mathbf{x}$. It is the measure of the heat conduction of the solid phase. For locally isotropic media, $\Lambda = \lambda \mathbf{I}$, where $\mathbf{I}$ is the identity tensor. Then, λ is called *the local thermal conductivity* or simply *the conductivity*. The thermal conductivity is considered as a scalar positive function $\lambda = \lambda(\mathbf{x})$ for locally isotropic materials and as a tensor function for locally anisotropic materials which in Cartesian coordinates has the form of the symmetric positively defined matrix:

$$\Lambda = \Lambda(\mathbf{x}) \begin{pmatrix} \lambda_{11}(\mathbf{x}) & \lambda_{21}(\mathbf{x}) & \lambda_{31}(\mathbf{x}) \\ \lambda_{12}(\mathbf{x}) & \lambda_{22}(\mathbf{x}) & \lambda_{23}(\mathbf{x}) \\ \lambda_{13}(\mathbf{x}) & \lambda_{23}(\mathbf{x}) & \lambda_{33}(\mathbf{x}) \end{pmatrix} \tag{5.2}$$

For Λ depending on the temperature, i.e., $\Lambda = \Lambda(\mathbf{x}, T)$ we deal with *nonlinear heat conduction*.

Sometimes the thermal resistance r is introduced as $r = \lambda^{-1}$ ($R = \Lambda^{-1}$), where the power -1 denotes the reciprocal whenever λ is a function (the matrix inverse whenever Λ is a matrix).

Assuming the presence of sources and sinks with intensity $f(\mathbf{x})$, we get the following relation $\nabla \cdot \mathbf{q} = f$ in D. If a medium does not contain sources or sinks, the heat flux satisfies the so-called *free divergence equation*:

$$\nabla \cdot \mathbf{q} = 0 \tag{5.3}$$

or in Cartesian coordinates $\frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} + \frac{\partial q_3}{\partial x_3} = 0$

Substituting (5.3) into (5.1), we obtain the elliptic equation:

$$\nabla \cdot (\Lambda \nabla T) = 0 \tag{5.4}$$

Laplace Equation In the case of isotropic homogeneous material, the conductivity $\lambda(\mathbf{x})$ is a constant. Then, (5.4) becomes the *Laplace equation*:

$$\nabla^2 T = 0 \tag{5.5}$$

i.e., T is a harmonic function in D. The constitutive relation (5.1) means in this case that the flux $\mathbf{q}(\mathbf{x})$ has a potential (in other words, the vector field $\mathbf{q}(\mathbf{x})$ is a potential one, or the considered physical system is conservative; e.g. Refs. [22,23]).

Geometry The domain D occupied by the medium is supposed to be an arbitrary open set in $\mathbb{R}^M$, $M = 3$(or $M = 2$). Usually it is supposed that D consists of a finite or denumerable collection of connected components. One of the components Ω, called the matrix or host material, contains other components as inclusions or pores.

If the boundary surfaces (or curves) are simply smooth, then two continuous families of normal vectors can be chosen. Each one generates an orientation on the surface (on the curve). Most common is to choose an orientation generated by outward unit normal vectors $\mathbf{n}$ to $\partial\Omega$, understanding such an orientation as positive and an opposite orientation as a negative one. Such a definition can be extended to domains with piecewise smooth boundaries.

The level of smoothness of the boundary can also be prescribed. For simplicity, it is usually supposed that $\partial\Omega$ consits of piecewise Lyapunov's (or $\mathcal{C}^{1,\alpha} - (0 < \alpha \leq 1)$) surfaces or curves. It means that they have a tangent plane (a tangent line) everywhere besides a finite number of smooth curves on a surface (finite number of points on a curve), and the corresponding field of normal vectors is Hölder continuous with respect to the spatial variable on a surface (on a curve). Therefore, corner (wedge) points can arise on the boundary surfaces (curves). It usually brings additional difficulties to attack problems (e.g., Refs. [24–26]).

Sometimes it is important to model the conducting medium by a certain infinite domain. Different compactifications can be applied in this case. For instance, in a 2D situation ($M = 2$) it is convenient to understand ∞ as the unique point extension of the complex plane $\mathbb{C}$ (i.e. as a north pole on the Riemann sphere $S^2 = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$) (e.g., Ref. [27]). In this case, the point ∞ can be either boundary point of a domain D or its internal point (on the sphere $\hat{\mathbb{C}}$). Another possibility is to consider several infinite points.

Spaces Looking for a classical solution, we need to prescribe certain smoothness of these solutions on the boundary. Let us recall definitions of the most standard spaces of smooth and piecewise smooth functions on a connected subset $X \subset \mathbb{R}^M$ (in particular, on each connected component of $\partial\Omega$).

It is said that the family $\mathcal{C}(X) = \{ f : X \to \mathbb{C}(\mathbb{R}) : f \text{ is continuous on } X \}$ forms a linear space of *continuous functions*. For X being either a closed surface or a closed curve, $\mathcal{C}(X)$ becomes a Banach space under the norm $\|f\|_c = \sup_{\mathbf{x}\in X}|f(\mathbf{x})|$. The spaces $\mathcal{H}^\alpha(X)$ containing *Hölder-continuous functions* are introduced by the following condition:

$$\mathcal{H}^\alpha(X) = \{ f \in \mathcal{C}(X) : \exists\, C > 0, \quad |f(\mathbf{x}_1) - f(\mathbf{x}_2)| < C|\mathbf{x}_1 - \mathbf{x}_2|^\alpha, \\ \forall\, \mathbf{x}_1, \mathbf{x}_2 \in X, 0 < \alpha \leq 1 \}$$

(here $|\mathbf{x}_1 - \mathbf{x}_2|$ means the Euclidean distance between two points of $X \in \mathbb{R}^M$). These spaces are called *Hölder spaces*. They are linear subspaces of $\mathcal{C}(X)$ (the following notations for them are also commonly used: $Lip_\alpha(X)$, $\mathcal{C}^{0,\alpha}$). Again for X being either a closed surface or a closed curve, $\mathcal{H}^\alpha(X)$ becomes a Banach space with the following norm:

$$||f||_{\alpha} = ||f||_{\mathcal{C}} + \sup_{\mathbf{x}_1,\mathbf{x}_2 \in X, \mathbf{x}_1 \neq \mathbf{x}_2} \frac{||f(\mathbf{x}_1) - f(\mathbf{x}_2)||}{|\mathbf{x}_1 - \mathbf{x}_2|^{\alpha}} =: ||f||_{\mathcal{C}} + h(f;\alpha)$$

For X being a nonclosed smooth surface or curve, a piecewise smooth surface or curve one can introduce *weighted Hölder spaces*:

$$\mathcal{H}^{\alpha}(X;\rho) = \{ f : \exists\, f_0, \quad f(\mathbf{x}) = f_0(\mathbf{x})\rho(\mathbf{x}), \quad f_0 \in \mathcal{H}^{\alpha}(X) \}$$

with a given weight function ρ (for instance, $\rho(\mathbf{x}) = \prod_{l=1}^{n} |\mathbf{x} - \mathbf{x}_l|^{\beta_l}$, where $\beta_l \in \mathbb{R}$).

To introduce the spaces of differentiable functions in $\mathbb{R}^M$, it is convenient to use the notion of multi-index. Let, for example, $\mathbf{x} = (x_1, x_2, x_3) \in X \subset \mathbb{R}^3$, $\quad f : X \to \mathbb{C}(\mathbb{R})$. Denote by $\partial_j f = \frac{\partial f}{\partial x_j}$ the derivative of f with respect to j-th variable. The vector $k = (k_1, k_2, k_3) \in \mathbb{Z}_+^3$ is called multi-index, and $|k| = k_1 + k_2 + k_3$ denotes its length. By definition, k-th (partial) derivative of f is equal to $\partial^k f = \frac{\partial^{|k|} f}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^{k_3}}$. Then, $\mathcal{C}^m(X)$, $m \in \mathbb{N}$, is a space of all functions $f : X \to \mathbb{C}(\mathbb{R})$ such that $f(\mathbf{x})$ and $\partial^k f, |k| \leq m$, are continuous on X. We have to note that for X being a smooth surface or a smooth curve the derivatives can be taken only in the tangent direction to X.

The collection of *Schauder spaces* is defined in the following way ($m \in \mathbb{N}$, $0 < \alpha \leq 1$):

$$\mathcal{C}^{m,\alpha}(X) = \{ f \in \mathcal{C}^m(X) : \exists\, C > 0, \quad |\partial^k f(\mathbf{x}_1) - \partial^k f(\mathbf{x}_2)| < C|\mathbf{x}_1 - \mathbf{x}_2|^{\alpha}, \\ \forall\, \mathbf{x}_1, \mathbf{x}_2 \in X, \quad |k| = m \}.$$

They are Banach spaces for X being a smooth surface or a smooth curve under the following norms:

$$||f||_{m,\alpha} = \sum_{|k|=0}^{m} ||\partial^h f||_{\mathcal{C}} + \sum_{|k|=m} h(\partial^k f;\alpha)$$

Finally, the collection of all *infinitely differentiable functions* is denoted by $\mathcal{C}^{\infty}(X)$:

$$\mathcal{C}^{\infty}(X) = \{ f \in \mathcal{C}(X) : \forall\, k \in \mathbb{Z}_+^3, \ \forall\, \mathbf{x} \in X, \ \exists\, \partial^k f(\mathbf{x}) \}$$

It should be noted that we can consider $\mathbb{R}^2$ to be isometric to $\mathbb{C}$. Thus, in this case in all the above definitions the partial derivatives can be replaced by the derivatives with respect to complex variable. These definitions are stronger than those with partial derivatives with respect to two real variables [27].

Let Γ be a simple closed curve in $\mathbb{C}$, $X = \operatorname{int} \Gamma$. The set of all continuous functions $\mathcal{C}(\Gamma)$ analytically extended into X is denoted by $\mathcal{C}_A(\Gamma)$ (or $\mathcal{C}_A(X)$). It is a Banach space under supremum norm on *cl* X. Analogous definitions (under corresponding norms in *cl* X) are used for the spaces $\mathcal{C}_A^{m,\alpha}(\Gamma) = \mathcal{C}_A^{m,\alpha}(X)$, $\mathcal{C}_A^{\infty}(\Gamma) = \mathcal{C}_A^{\infty}(X)$.

In many problems of mathematical physics it is not sufficient to deal only with the classical solutions corresponding to differential equations. One of the most suitable generalizations of the above introduced spaces are the so-called *Sobolev spaces* (e.g., Ref. [28]). The main idea in the construction of these spaces is the use of the notion of *weak derivatives*.

Let $X \subset \mathbb{R}^M$ be an open connected set. We denote by $L_p(X)$, $1 \le p < \infty$, the set of all (Lebesgue-) measurable functions $f : X \to \mathbb{C}(\mathbb{R})$ such that

$$||f||_p = \left(\int_X |f(\mathbf{x})|^p \mathrm{d}\mathbf{x} \right)^{1/p} < \infty$$

It is a Banach space with respect to the norm $||f||_p$. It is said that $X' \subset X \subset \mathbb{R}^M$ is a strictly interior subdomain of X, if $\bar{X}' \subset X$ (the corresponding notation is $X' \subset\subset X$). The space $L_{p,loc}(X)$, $1 \le p < \infty$, is the set of all (Lebesgue-) measurable functions $f : X \to \mathbb{C}(\mathbb{R})$ such that $\int_{X'} |F(x)|^p \mathrm{d}\mathbf{x} < \infty$ for any bounded strictly interior domain $X' \subset\subset X$. Denote also by $C_0^\infty(X)$ the class of all compactly supported infinitely differentiable functions $f : X \to \mathbb{C}(\mathbb{R})$.

Let now k be a multi-index, the functions $f, g \in \mathrm{L}_{1,loc}(X)$, and the following relation be satisfied:

$$\int_X f(\mathbf{x}) \partial^k \eta(\mathbf{x}) \mathrm{d}\mathbf{x} = (-1)^{|k|} \int_X g(\mathbf{x}) \eta(\mathbf{x}) \mathrm{d}\mathbf{x}$$

for all functions $\eta \in C_0^\infty(X)$. Then, g is called the *weak derivative* of f. It is denoted by the same symbol as before, namely $g = \partial^k f$.

The Sobolev space $W^{s,p}(X)$, $1 \le p < \infty$, $s \in \mathbb{Z}_+$, is the set of all functions $f \in L_p(X)$ for which there exist weak derivatives $\partial^k f$ for any k, $|k| < s$, such that $\partial^k f \in L_p(X)$, $\forall |k| \le s$. Endowed by the norm $||f||_{s,p,X} = \sum_{|k| \le s} ||\partial^k f||_p$, $W^{s,p}(X)$ becomes a Banach space. In the special case $p = 2$, the space $W^{s,2}(X)$ is denoted by $H^s(X)$. This space is a Hilbert space with the inner product defined by the relation

$$\langle f, g \rangle_{s,X} = \sum_{|k| \le s} \langle \partial^k f, \partial^k g \rangle L_2(X) = \sum_{k| \le s} \int_X \partial^k f(\mathbf{x}) \overline{\partial^k g(\mathbf{x})} \mathrm{d}\mathbf{x}$$

The set $C_0^\infty(X)$ is a dense subset of $H^1(\mathbb{R}^M)$. If $X \ne \mathbb{R}^M$ then the closure of $C_C^\infty(X)$ in H^1-norm is a subspace of $H^1(X)$ denoted by $H_0^1(X)$.

The set of all linear continuous functional $H_0^1(X)$ with respect to inner form in $H_0^1(X)$:

$$\langle f, g \rangle_{1,X} = \left(\int_X f(\mathbf{x}) \overline{g(\mathbf{x})} \mathrm{d}\mathbf{x} + \sum_{|k|=1} \int_X \partial^k f(\mathbf{x}) \overline{\partial^k g(\mathbf{x})} \mathrm{d}\mathbf{x} \right)$$

is denoted by $H^{-1}(X)$.

Let Q be a parallelepiped in $\mathbb{R}^M$, $M=2,3$, then $C^{m,\alpha}_{A,per}(\partial Q)=C^{m,\alpha}_{A,per}(Q)$, $C^{\infty}_{A,per}(\partial Q)=C^{\infty}_{A,per}(Q)$, $W^{s,p}_{per}(Q)$, $H^{s}_{per}(Q)$ are subspaces of $C^{m,\alpha}_{A}(Q)$, $C^{\infty}_{A}(Q)$, $W^{s,p}(Q)$, $H^s(Q)$, respectively, containing those functions which possess periodic extension from the set Q to the space $\mathbb{R}^M$.

5.2.2
Boundary Value Problems

Let us present different types of boundary value problem for heat conduction in composites and porous media (i.e. boundary value problems for equations (5.4) and 5.5)).

To be more precise, we formulate these problems in the case of composites consisting of the matrix (which is a multiply connected domain Ω in $\mathbb{R}^2$ with outer boundary curve Γ) and of n inclusions D_k, $k=1,\ldots,n$, encircled by smooth closed surfaces (curves) $L_k=\partial D_k$. It is convenient to use the notation $\Lambda(\mathbf{x})$ for the conductivity tensor for the material occupied by the host domain Ω, and $\Lambda_k(\mathbf{x})$, $k=1,\ldots,n$, for the conductivity tensors for the material occupied by the corresponding inclusions (see Section 5.2.1).

We suppose that either $M=3$ or $M=2$, just by making corresponding remarks when these situations differ essentially. According to the above presented description of the orientation on the boundary of Ω, we will denote by $T(\mathbf{t})$ the boundary values of the temperature distribution on Γ, and the boundary limits on L_k of the temperature from the domain Ω and domains D_k by $T^+(\mathbf{t})$, $T_k^-(\mathbf{t})$, respectively:

$$T(\mathbf{t})=\lim_{\mathbf{x}\to\mathbf{t}\in\Gamma,\mathbf{x}\in\Omega}T(\mathbf{x});\quad T^+(\mathbf{t})=\lim_{\mathbf{x}\to\mathbf{t}\in L_k,\mathbf{x}\in\Omega}T(\mathbf{x}),\ k=1,\ldots,n;$$
$$T_k^-(\mathbf{t})=\lim_{\mathbf{x}\to\mathbf{t}\in L_k,\mathbf{x}\in D_k}T(\mathbf{x})$$

The given temperature distribution $f(\mathbf{t})$ on the outer boundary Γ leads to the Dirichlet condition on Γ:

$$T(\mathbf{t})=f(\mathbf{t}),\quad \mathbf{t}\in\Gamma \tag{5.6}$$

If the outer boundary constitutes the ideal thermal isolator (i.e. there is no heat exchange between the composite and the medium outside of it), then we arrive at the homogeneous Neumann condition

$$\frac{\partial T}{\partial\mathbf{n}}(\mathbf{t})=0,\quad \mathbf{t}\in\Gamma \tag{5.7}$$

If there is heat transfer through the outer boundary when the normal heat flux $\mathbf{q}\cdot\mathbf{n}$ is known at the outer surface, then condition (5.7) should be replaced by a more complicated one (see Eq. (5.1))

$$\Lambda\nabla T\cdot\mathbf{n}(\mathbf{t})=g(\mathbf{t}),\quad \mathbf{t}\in\Gamma \tag{5.8}$$

Here, $\Lambda\nabla T$ denotes that the matrix Λ is multiplied by the vector $\nabla\Lambda$. Further, the scalar product of the vectors $\Lambda\nabla T$ and $\mathbf{n}$ is calculated.

Instead of Eq. (5.8), the heat transfer satisfying Newton's law can be considered at the boundary

$$\lambda \frac{\partial T}{\partial \mathbf{n}}(\mathbf{t}) + \gamma T(\mathbf{t}) = h(\mathbf{t}), \quad \mathbf{t} \in \Gamma \tag{5.9}$$

It is also called the third type boundary value problem.

5.2.3 Conjugation Problem

Other types of condition arise on internal components of the boundary of Ω, i.e. on the matrix–inclusions (host–pores) interface. The most natural are continuity of the temperature and of the heat flux. For simplicity, hereafter the scalar conductivity is considered. Then, they have the following form

$$T^+(\mathbf{t}) = T_k^-(\mathbf{t}), \quad \lambda \frac{\partial T^+}{\partial \mathbf{n}}(\mathbf{t}) = \lambda_k \frac{\partial T_k^-}{\partial \mathbf{n}}(\mathbf{t}), \quad \mathbf{t} \in L_k (k = 1, \ldots, n) \tag{5.10}$$

They are known as the *perfect contact* or *transmission* conditions. It is also natural to assume that the temperature distribution and the normal heat flux have jumps along a part of the matrix–inclusions interface. In this case, conditions (5.10) have the following form:

$$T^+(\mathbf{t}) - T_k^-(\mathbf{t}) = h_k(\mathbf{t}), \quad \lambda \frac{\partial T^+}{\partial \mathbf{n}}(\mathbf{t}) - \lambda_k \frac{\partial T_k^-}{\partial \mathbf{n}}(\mathbf{t}) = g_k(\mathbf{t}), \quad \mathbf{t} \in L_k (k = 1, \ldots, n) \tag{5.11}$$

where h_k, g_k are given functions on L_k.

If at least a part of the matrix–inclusions interface consists of poorly conducting material then we have to replace the first series of the above conditions by a more complicated one, namely, we have the following problem:

$$\lambda \frac{\partial T^+}{\partial \mathbf{n}}(\mathbf{t}) + \gamma_k (T^+(\mathbf{t}) - T_k^-(\mathbf{t})) = 0, \quad \lambda \frac{\partial T^+}{\partial \mathbf{n}}(\mathbf{t}) = \lambda_k \frac{\partial T_k^-}{\partial \mathbf{n}}(\mathbf{t}),$$
$$\mathbf{t} \in L_k (k = 1, \ldots, n) \tag{5.12}$$

The coefficients γ_k^{-1} introduced in Eq. (5.12) and known as the Kapitza resistances [3,29]. The limit cases $\gamma_k = 0$, and $\gamma_k = \infty$ were discussed in Ref. [16].

A special problem can be also considered, namely with the boundary conditions given on the exterior boundary and the domains D_k occupied by an ideal conductor ($\lambda_k = \infty$). In this case, we arrive at the modified Dirichlet problem [30]

$$T(\mathbf{t}) = t_k, \quad \mathbf{t} \in L_k \tag{5.13}$$

where t_k are undetermined constants which have to be found in the solution to the problem.

5.2.4
Complex Potentials

The aim of this subsection is to rewrite equations as well as boundary value problems for heat conduction in composites (or in porous media) in terms of complex analysis. Thus, we have studied here only the two-dimensional situation ($M = 2$) considering the corresponding domains as domains on the complex plane $\mathbb{C}$. In this case, it is supposed that the heat flux is spreading in a direction orthogonal to the cylinder in which parallel cylindrical inclusions are implemented. The base of the cylinder is a multiply connected domain Ω, and the bases of the inclusions are domains D_k. There is also another statement of the 2D problem when a thick plate with isolated sides is considered.

First, we consider the limit cases when Ω is occupied by a conducting material and on the boundary of which one of the boundary conditions (5.6), (5.8) and (5.9) are given. Consider the Dirichlet problem (5.6). It is known that each harmonic function in a simply connected domain is the real part of a complex potential. If a function $T(x, y)$ is harmonic in a multiply connected domain Ω then it can be expressed as

$$T(z) = \mathrm{Re}\left[\Phi(z) + \sum_{k=1}^{n} A_k \ln(z - z_k)\right], \quad z = x_1 + ix_2 \in \Omega \tag{5.14}$$

according to the decomposition theorem [16]. Here, the function $\Phi(z)$ is analytic and single-valued in Ω, and A_k are real numbers. If we assume that $\infty \in \Omega$, D_k ($k = 1, 2, \ldots, n$) are connected components of the complement of Ω to $\mathbb{C}$, and z_k are points in D_k, then the connectivity of Ω is equal to $n - 1$ and

$$\sum_{k=1}^{n} A_k = 0 \tag{5.15}$$

Substituting $T(z)$ from Eq. (5.14) in Eq. (5.6), we arrive at the boundary value problem with respect to $\Phi(z)$. The constants A_k have also to be determined. One can find a discussion of this problem for multiply connected domains in Ref. [30] and a complete solution to this problem for any circular multiply connected domain in Ref. [16]. A similar argument can be applied to the problems (5.7) and (5.9).

Consider now the modified Dirichlet problem (5.13). In this case, instead of (5.14) we have $T(z) = \mathrm{Re}\,\Phi(z)$. However, the undetermined constants t_k are included in the boundary condition

$$\mathrm{Re}\,\Phi(t) = t_k, \quad t \in L_k \quad (k = 1, 2, \ldots, n)$$

We also suppose (again for simplicity) that the materials inside matrix and inclusions are isotropic and homogeneous, which means the constancy of conductivity coefficients λ, λ_k, $k = 1, \ldots, n$. Therefore, the temperature T is a harmonic function in the domains Ω and D_k, $k = 1, \ldots, n$ (i.e. satisfies in these domains the Laplace equation 5.5).

Let T, and T_k be temperature distributions in Ω and D_k, $k = 1, \ldots, n$, respectively, continuously differentiable up to the boundaries of these domains satisfying Eq. (5.5). Suppose that the perfect contact relations (5.10) are valid on each curve $L_k = \partial D_\mathrm{k}$, $k = 1, \ldots, n$. Then, one can introduce functions

$$\varphi(z) = T(z) + iV(z), \quad z \in \Omega, \quad \varphi_k(z) = \frac{\lambda + \lambda_k}{2\lambda}(T_k(z) + iV_k(z)),$$
$$z \in D_k, \quad k = 1, \ldots, n \tag{5.16}$$

which are analytic in Ω, D_k, respectively, continuously differentiable in the closures of the considered domains. In fact (e.g. Ref. [22]), the function $\varphi(z)$ is in general a multivalued analytic function since Ω is a multiply connected domain. But in our case, $T(z)$ possesses [31] a unique harmonic extension up to the function, harmonic in a simply connected domain $D = \Omega \bigcup_{k=1}^{n} L_k \bigcup_{k=1}^{n} D_k$ due to the first relations in Eq. (5.10). Therefore, due to the uniqueness of analytic continuation $\varphi(z)$ is a single-valued analytic function in Ω as the restriction of the corresponding function defined on D.

In order to represent the boundary conditions (5.10) in the complex form, we write the normal and tangent derivatives on a fixed curve L_k:

$$\frac{\partial}{\partial \mathbf{n}} = n_1 \frac{\partial}{\partial x_1} + n_2 \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial \mathbf{s}} = -n_2 \frac{\partial}{\partial x_1} + n_1 \frac{\partial}{\partial x_2} \tag{5.17}$$

where the normal vector $\mathbf{n}$ is identified with the complex number, $\mathbf{n} = n_1 + in_2$, the tangent vector $\mathbf{s} = n_2 - in_1$, and $z = x_1 + ix_2$. By applying the second operator of Eq. (5.17) to the first condition of (5.10), we obtain

$$-n_2 \frac{\partial T^+}{\partial x_1} + n_1 \frac{\partial T^+}{\partial x_2} = -n_2 \frac{\partial T_k^-}{\partial x_1} + n_1 \frac{\partial T_k^-}{\partial x_2} \tag{5.18}$$

The second condition of Eq. (5.10) can be rewritten in a similar way

$$\lambda n_1 \frac{\partial T^+}{\partial x_1} + \lambda n_2 \frac{\partial T^+}{\partial x_2} = \lambda_k n_1 \frac{\partial T_k^-}{\partial x_1} + \lambda_k n_2 \frac{\partial T_k^-}{\partial x_2} \tag{5.19}$$

We introduce new complex potentials in the domains Ω and D_k, respectively,

$$\psi = \frac{\partial \varphi}{\partial z} = \frac{\partial T}{\partial x_1} - i\frac{\partial T}{\partial x_2}, \quad \psi_k = \frac{\lambda_k + \lambda}{2\lambda}\left(\frac{\partial \varphi_k}{\partial z}\right) = \frac{\lambda_k + \lambda}{2\lambda}\left(\frac{\partial T_k}{\partial x_1} - i\frac{\partial T_k}{\partial x_2}\right) \tag{5.20}$$

Then, substituting these relations into Eqs. (5.18) and (5.19) and excluding $\overline{\psi^+}$, we arrive at the following conjugation condition:

$$\psi^+(t) = \psi_k^-(t) + \rho_k \overline{\mathbf{n}^2 \psi_k^-(t)}, \quad t \in L_k \tag{5.21}$$

where

$$\rho_k = \frac{\lambda_k - \lambda}{\lambda_k + \lambda} \tag{5.22}$$

is a contrast parameter introduced by Bergman [32,33]. Integrating Eq. (5.21) along L_k with constant of integration equal to zero [16], we obtain the following boundary value problem for analytic functions in a multiply connected domain, namely, for the complex potentials φ, φ_k, $k=1,\ldots,n$:

$$\varphi^+(t) = \varphi_k^-(t) - \rho_k\overline{\varphi_k^-(t)}, \quad t \in L_k \tag{5.23}$$

This problem is a special case of so-called $\mathbb{R}$-linear conjugation problem (Markushevich's problem) (see Refs. [34,35] for the description of qualitative results concerning the solvability of $\mathbb{R}$-linear conjugation problem with arbitrary coefficients).

If at least one of the first conditions in Eq. (5.10) is replaced by a non-zero jump condition, i.e. we have Eq. (5.11), then one can proceed in a similar way as before. We introduce the complex potentials by formulas (5.16). If h_k are smooth enough, e.g. $h_k \in C^{1,\alpha}(L_k)$, then one can find (single-valued) analytic in D_k functions $h_k^-(z)$ satisfying the following boundary conditions (the so-called Schwarz boundary value problem):

$$\mathrm{Re}\, h_k^-(t) = h_k(t), \quad t \in L_k \tag{5.24}$$

Then, the first conditions of Eq. (5.11) can be rewritten in the form $T^+(t) - \tilde{T}_k^-(t) = 0$, $t \in L_k$, $k = 1, \ldots, n$, where $\tilde{T}_k^-(z) = T_k^-(z) + \mathrm{Re}\, h_k^-(z)$, $z \in D_k$. A similar argument can be applied to the second condition (5.11). Thus, the last relations give the single-valuedness of the potential $\varphi^+(z)$ in Ω. Then by using the above introduced complex potentials (5.16), we deduce that the only difference is that at least one of the Eqs. (5.18) is in this case inhomogeneous. As a result, we will have the following boundary value problem with non-zero inhomogeneous term $c_k(t)$ on at least one curve L_k:

$$\varphi^+(t) = \varphi_k^-(t) - \rho_k\overline{\varphi_k^-(t)} + c_k(t), \ t \in L_k \tag{5.25}$$

Exact calculation of the inhomogeneous term can be easily done. It does not have much influence on further analysis.

Application of the same arguments to Eq. (5.12) yields an $\mathbb{R}$-linear conjugation problem with derivatives. Let us assume for simplicity that the inclusions are circular cylinders, i.e. $D_k = \{z \in \mathbb{C} : |z - a_k| < r_k\}$, $k = 1, \ldots, n$. Then, the problem (5.12) becomes

$$\varphi^+(t) = \varphi_k^-(t) - \rho_k\overline{\varphi_k^-(t)} + \mu_k(t - a_k)(\varphi_k^-)'(t) + \mu_k \frac{r_k^2}{t - a_k}\overline{(\varphi_k^-)'(t)}, \quad |t - a_k| = r_k \tag{5.26}$$

where $\mu_k = \frac{1+\rho_k}{2 r_k \gamma_k}$.

Finally, we have to rewrite the boundary conditions on the outer boundary Γ in the complex form too. The simplest is to reformulate the Dirichlet condition (5.6). We first solve the outer Schwarz boundary value problem

$$\operatorname{Re} f_0(t) = f(t), \quad t \in \Gamma \tag{5.27}$$

with respect to the function $f_0(z)$ analytic outside of the whole domain $D = \Omega \bigcup_{k=1}^{n} L_k \bigcup_{k=1}^{n} D_k$, i.e. in *ext D*. Then by introducing an auxiliary unknown function $\varphi_0(z)$ analytic in *ext D*, $\varphi_0(\infty) = 0$ and using the same complex potential $\varphi(z)$ for Ω, we rewrite Eq. (5.6) in the form of $\mathbb{R}$-linear conjugation problem:

$$\varphi^+(t) = \varphi_0(t) - \overline{\varphi_0(t)} + f_0(t), \quad t \in \Gamma \tag{5.28}$$

A similar approach is used for the Neumann problem (5.7) (for complex potential ψ) [16].

Problems (5.10) and (5.11) were discussed for arbitrary multiply connected domains in Ref. [30] and solved explicitly for any multiply connected domain in Ref. [16].

5.2.5
Periodic Problems

Boundary value problems for harmonic and analytic functions discussed in the previous sections are also stated in classes of periodic functions. As it follows from the theory of homogenization [17,18,20], such problems are the basis for rigorous definition of the effective conductivity tensor. In the present section, we state periodic problems.

Consider a lattice $\mathcal{Q}$ which is defined by three fundamental translation vectors **a**, **b**, **c**. Hereafter, we consider orthogonal lattices, i.e. the vectors **a**, **b**, **c** generate an orthogonal system of coordinates. For definiteness, we put $\mathbf{a} = (\alpha,0,0)$, $\mathbf{b} = (0,\beta,0)$, $\mathbf{c} = (0,0,\gamma)$. Let $\mathbf{m} = (m_1,m_2,m_3)$ denote vectors with integers components. Introduce the zero-th cell (representative cell) $Q \equiv Q_0 := \{\mathbf{x} = (x_1, x_2, x_3) : -\alpha/2 < x_1 < \alpha/2, -\beta/2 < x_2 < \beta/2, -\gamma/2 < x_3 < \gamma/2\}$. The lattice $\mathcal{Q}$ generates the cells $Q_{\mathbf{m}}: \{\mathbf{x} \in \mathbb{R}^3 : (x_1 - \alpha m_1, x_2 - \beta m_2, x_3 - \gamma m_3) \in Q\}$. Without loss of generality, it is assumed that the volume of Q is equal to unity, i.e. $|Q| = \alpha\beta\gamma = 1$. Consider mutually disjoint domains D_k $(k = 1,2,\ldots,n)$ lying in the zero-th cell Q. Let D be the complement of the closure of all D_k to Q.

Let a constant external gradient (q_1,q_2,q_3) be applied to the material. Then, the temperature distribution satisfies the conjugation condition (5.10) and the quasi-periodicity relations with respect to the lattice $\mathcal{Q}$:

$$\begin{aligned} T(x_1 + \alpha, x_2, x_3) - T(x_1, x_2, x_3) &= q_1 \\ T(x_1, x_2 + \beta, x_3) - T(x_1, x_2, x_3) &= q_2 \\ T(x_1, x_2, x_3 + \gamma) - T(x_1, x_2, x_3) &= q_3 \end{aligned} \tag{5.29}$$

Theorem 5.2.1 [17,20] *The problem (5.10), (5.29) with fixed* q_1, q_2 *and* q_3 *has a unique solution in* $H^1(Q)$ *up to an arbitrary additive real constant.*

In this subsection, the condition of the perfect contact (5.10) is considered. In a similar way, other boundary and conjugation problems can be stated. Frequently, instead of the quasi-periodicity conditions (5.29) boundary conditions are introduced for symmetric problems as follows. Let the cell Q be symmetric with respect to the coordinate axes, i.e. the inclusions are symmetrically located in Q. Let the external gradient $(c_1,0,0)$ be applied to the periodic material. Then, the the heat flux is symmetric with respect to the planes $x_2=\beta/2+m$ and $x_3=\gamma/2+m$, where $m=0,\pm1,\pm2,\ldots$. In particular, this symmetry yields the Neumann condition:

$$\frac{\partial T}{\partial x_2}(x_1,\pm\beta/2,x_3)=0,\quad \frac{\partial T}{\partial x_3}(x_1,x_2,\pm\gamma/2)=0 \tag{5.30}$$

The first quasi-periodicity conditions (5.29) for the temperature distribution yield the periodicity conditions for $\frac{\partial T}{\partial x_1}$ on the planes $x_1=\pm\alpha/2$. Then, T is a constant on $x_1=\pm\alpha/2$ and one can take the Dirichlet conditions:

$$T(\pm\alpha/2,x_2,x_3)=\pm q_1/2 \tag{5.31}$$

Therefore, the quasi-periodic problems (5.10), (5.29) becomes the mixed problem (5.10), (5.30), (5.31) for the domain Q. Such an approach is applied in Section 5.5.

Following Section 5.2.4, consider 2D quasi-periodic problems by the method of complex potential. In this case, it is convenient to express the fundamental translation vectors by the complex numbers α and $i\beta=i/\alpha$. The problem (5.10), (5.29) becomes the $\mathbb{R}$-linear conjugation problem (5.23) with the quasi-periodicity conditions

$$\varphi(z+\alpha)-\varphi(z)=q_1+id_1,\quad \varphi(z+i\alpha^{-1})-\varphi(z)=q_2+id_2 \tag{5.32}$$

where q_1 and q_2 are given real constants, d_1 and d_2 are undetermined real constants which should be found. The function $\varphi(z)$ is analytic in D, D_k, continuously differentiable in the closure of the considered domains.

The problem (5.10), (5.32) can be considered as an $\mathbb{R}$-linear conjugation problem on the torus represented by the cell Q. It can also be considered as an $\mathbb{R}$-linear conjugation problem for an infinitely connected domain bounded by ∂D_k+m ($m=m_1\alpha+im_2\alpha^{-1}$, and m_1 and m_2 run over integers).

Along similar lines, the periodic $\mathbb{R}$-linear conjugation problem can be stated in the form (5.21). Additionally, $\psi(z)$ satisfies the periodicity conditions

$$\psi(z+\alpha)-\psi(z)=0,\quad \psi(z+i\alpha^{-1})-\psi(z)=0 \tag{5.33}$$

Theorem 5.2.2 ([77]) *The problem (5.10), (5.32) has a unique solution in $C_A^{2,\alpha}(Q)$ up to an arbitrary additive complex constant, say $C + i\gamma$. This solution is related to the solution $T(z)$ of the 2D problem (5.10), (5.29) by the formulas*

$$\varphi(z)\begin{cases} \frac{\lambda_k + 1}{2}(T(z) + iV(z)), & z \in D_k, \quad k = 1, 2, \ldots, n \\ T(z) + iV(z), & z \in D \end{cases} \tag{5.34}$$

where $V(z)$ is a harmonic conjugate to $T(z)$. The constant C is the additive arbitrary real constant from the general solution of the 2D problem (5.10), (5.29).

Theorem 5.2.3 [36] *General solution of the homogeneous problem (5.21), (5.33) in $C_A^{2,\alpha}(Q)$ has the form*

$$\psi(z) = q_1\psi^{(1)}(z) + q_2\psi^{(2)}(z), \tag{5.35}$$

where $\psi^{(1)}(z)$ and $\psi^{(2)}(z)$ are partial linearly independent solutions of the problem (5.21), (5.33), q_1 and q_2 are arbitrary real constants.

This solution can be fixed by the relation $\psi^{(j)}(z) = (\varphi^{(j)})'(z)$, where $\varphi^{(j)}(z)$ are as in Theorem 5.2.2; $\varphi^{(1)}(z)$ is a solution of the problem (5.10), (5.32) with $q_1 = 1$, $q_2 = 0$ and $\varphi^{(2)}(z)$ is a solution of the corresponding 2D problem with $q_1 = 0$, $q_2 = 1$.

The decomposition theorem (see representation (5.14)) on the torus generated by the lattice $\mathcal{Q}$ was described in Ref. [37]. Any function $T(z)$ which is harmonic in Ω and doubly periodic can be written as

$$T(z) = \mathrm{Re}\left\{\varphi(z) + \sum_{k=1}^{n} A_k[\ln\sigma(z - a_k) + a_k\zeta(z - a_k)]\right\}, \quad z \in D \tag{5.36}$$

where σ and ζ are Weierstrass functions (see Section 5.6.1); A_k are real constants satisfying relation (5.15). The function $\varphi(z)$ is analytic in Ω and quasi-periodic. The choice of a branch of the logarithm does not impact on the value $T(z)$ because we actually deal with $\mathrm{Re}\{\ln z\}$ in (5.36). One can choose an arbitrary brunch of $\ln(z - a_k)$ and suppose that the cut corresponding to this fixed brunch is doubly periodic and has no common points with D_m for each $m \neq k$. Using the representation (5.36), one can reduce periodic boundary value problems for harmonic functions to problems for analytic functions following Section 5.2.4.

5.3 Effective Conductivity Tensor

Although the notation *effective conductivity tensor* is intuitively clear for physicists and engineers, the rigorous mathematical definition of the effective conductivity tensor needs a certain theoretical justification. One of the possible ways for such a justification is the use of homogenization theory. Following Refs. [1,2,17,18,20], consider a periodic composite. Let the linear sizes of the periods be of order ε L, where L is the linear order of the sample D bounded by a simple closed curve Γ. Consider the Dirichlet problem [20, p.20 Russian edn] in $H_0^1(D)$

$$\nabla(\Lambda_\varepsilon(\mathbf{x})\nabla T_\varepsilon(\mathbf{x})) = 0 \tag{5.37}$$

$$T_\varepsilon(\mathbf{t}) = f(\mathbf{t}), \quad \mathbf{t} \in \Gamma \tag{5.38}$$

Let

$$\Lambda_\varepsilon(\mathbf{x})\nabla T_\varepsilon(\mathbf{x}) \rightharpoonup \hat{\Lambda}\nabla T_0 \text{ in } L_2(D) \tag{5.39}$$

where $\rightharpoonup$ means the weak convergence in $L_2(D)$, $\hat{\Lambda}$ is a constant tensor and T_0 is a solution of the Dirichlet problem

$$\nabla(\hat{\Lambda}\nabla T_0(\mathbf{x})) = 0, \quad T_0(\mathbf{t}) = f(\mathbf{t}), \quad \mathbf{t} \in \Gamma \tag{5.40}$$

Then, the tensor $\hat{\Lambda}$ is called the effective conductivity tensor. The homogenization theory justifies the existence of the weak limit (5.39) and the independence of the limit of the shape of Γ and boundary conditions. For instance, instead of the Dirichlet condition (5.38), the Neumann condition can be taken. Moreover, the homogenization theory implies that $\hat{\Lambda}$ can be calculated by the formula

$$\hat{\Lambda}\mathbf{q} = \langle \Lambda(\mathbf{x})\nabla T(\mathbf{x})\rangle \tag{5.41}$$

where $\langle F(\mathbf{x})\rangle$ denote the average over the cell Q

$$\langle F(\mathbf{x})\rangle = \frac{1}{|Q|}\int_Q F(\mathbf{x})d\mathbf{x} \tag{5.42}$$

$|Q|$ is the area of Q. The function $T(\mathbf{x})$ is a solution of the quasi-periodic problem (see Section 5.2.5):

$$\begin{aligned} &\nabla(\Lambda(\mathbf{x})\nabla T(x)) = 0, \quad \mathbf{x} \in Q \\ &T(x_1 + \alpha, x_2, x_3) - T(x_1, x_2, x_3) = q_1 \\ &T(x_1 x_2 + \beta, x_3) - T(x_1, x_2, x_3) \ = q_2 \\ &T(x_1, x_2, x_3 + \gamma) - T(x_1, x_2, x_3) = q_3 \end{aligned} \tag{5.43}$$

Here, $\mathbf{q} = (q_1,q_2,q_3)$ is the external flux. One can see that $\hat{\Lambda}$ is completely determined by (5.41) via solution to three problems (5.43) with $\mathbf{q} = (1,0,0)$, $\mathbf{q} = (0,1,0)$, $\mathbf{q} = (0,0,1)$.

In general, $\hat{\Lambda}$ is a symmetric positively defined tensor. It can be reduced to the diagonal form

$$\hat{\Lambda} = \begin{pmatrix} \hat{\lambda}_1 & 0 & 0 \\ 0 & \hat{\lambda}_2 & 0 \\ 0 & 0 & \hat{\lambda}_3 \end{pmatrix} \tag{5.44}$$

More precisely, there exists a coordinate system in which the tensor $\hat{\Lambda}$ has the diagonal form (5.44). The axes $x'_j (j = 1, 2, 3)$ of this new coordinate system are called *the principal axes*. The component $\hat{\lambda}_j$ $(j = 1,2,3)$ is called the conductivity in the x'_j-direction. The tensor ellipsoid, invariants of the tensor and other fundamental properties of tensors can be found in standard courses on tensor algebra (e.g. Ref. [38]).

The tensor $\hat{\Lambda}$ for macroscopically isotropic composites has the form

$$\hat{\Lambda} = \hat{\lambda}\mathbf{I}, \tag{5.45}$$

where $\mathbf{I}$ is the identity tensor, i.e. in this case $\hat{\lambda} := \hat{\lambda}_1 = \hat{\lambda}_2 = \hat{\lambda}_3$. The scalar $\hat{\lambda}$ is called the effective conductivity.

The variational statement of the problem implies the formula

$$\hat{\lambda}\mathbf{q} = \inf_{u \in H^1_{per}(Q)} \langle \lambda(\mathbf{x}) |\nabla u(\mathbf{x})|^2 \rangle = \langle \lambda(\mathbf{x}) |\nabla T(\mathbf{x})|^2 \rangle \tag{5.46}$$

Consider a 2D representative symmetric cell. Then, the periodicity cell problem is reduced to the mixed problem (5.10), (5.30), (5.31) for the domain Q. In this case, the following formula can used for the effective conductivity in the x_1-direction:

$$\hat{\lambda}_1 = \frac{4}{\alpha q_1} \int_{-\alpha/4}^{\alpha/4} \lambda\left(x_1, \frac{\beta}{4}\right) \frac{\partial T}{\partial x_1}\left(x_1, \frac{\beta}{4}\right) dx_1 \tag{5.47}$$

This formula expresses that the effective conductivity in the x_1-direction is equal to the average flux passing along the symmetry segment $x_2 = \frac{\beta}{4}, -\frac{\alpha}{4} < x_1 < \frac{\alpha}{4}$ divided by the jump of the temperature $\frac{q_1}{2}$ per the half-periodicity cell. Similar formulas take place for the conductivities $\hat{\lambda}_2$ and for corresponding coefficients in $\mathbb{R}^3$ (i.e. for 3D composites).

Consider now an application of the formula (5.41) to 2D matrix–inclusion composites. Using the functions $\psi^{(j)}(z)$ described in Theorem 5.2.3, we obtain the components of $\hat{\Lambda}$:

$$\begin{aligned} \hat{\lambda}_{11} - i\hat{\lambda}_{12} &= 1 + 2\sum_{k=1}^{n} \rho_k \int_{D_k} \varphi^{(1)}(z) dx_1 dx_2, \\ \hat{\lambda}_{22} + i\hat{\lambda}_{22} &= 1 + 2i\sum_{k=1}^{n} \rho_k \int_{D_k} \varphi^{(2)}(z) dx_1 dx_2 \end{aligned} \tag{5.48}$$

For macroscopically isotropic composites, we have

$$\hat{\lambda} = 1 + 2\sum_{k=1}^{n} \rho_k \int_{D_k} \varphi^{(1)}(z) dx_1 dx_2 \tag{5.49}$$

Consider the case when the inclusions D_k are disks $|z - a_k| < r_k$. Then, application of the mean value theorem to (5.49) yields

$$\hat{\lambda} = 1 + 2\sum_{k=1}^{n} \rho_k \pi r_k^2 \varphi^{(1)}(a_k) \tag{5.50}$$

Thus one can see that to determine $\hat{\lambda}$ we need only $\psi^{(1)}(a_k)$.

Remark 5.3.1 *The area of the representative cell $|Q|$ does not impact onto the effective conductivity. Hence, it can be normalized to unity.*

5.4 Review of Known Formulas

5.4.1 Laminates

Laminates are described by the local conductivity $\lambda(x_1)$ which depends only on one spatial variable in appropriate coordinates. Let the period of $\lambda(x_1)$ be equal to unity. Then, $\hat{\Lambda}$ has the form (5.44) where

$$\hat{\lambda}_1 = \left(\int_0^1 \frac{d\xi}{\lambda(\xi)}\right)^{-1}, \quad \hat{\lambda}_2 = \hat{\lambda}_3 = \int_0^1 \lambda(\xi) d\xi \tag{5.51}$$

Consider the case when $\lambda(x_1)$ takes values λ_1 and λ_2 with the probabilities (concentrations) ν_1 and ν_2, respectively. Then, (5.51) becomes

$$\hat{\lambda}_1 = \frac{1}{\frac{\nu_1}{\lambda_1} + \frac{\nu_2}{\lambda_2}}, \quad \hat{\lambda}_2 = \hat{\lambda}_3 = \nu_1\lambda_1 + \nu_2\lambda_2 \tag{5.52}$$

The general theory of laminates is presented in Refs. [2,3,39].

5.4.2 Clausius–Mossotti Approximation (CMA)

We begin with an analytical formula for the effective conductivity of macroscopically isotropic composites known from the eighteenth century. Consider a two-component macroscopically isotropic composite medium consisted of a collection of non overlapping identical balls of conductivity λ_1 imbedded into a host medium of conductivity λ. The effective conductivity $\hat{\lambda}$ of the considered inhomogeneous medium is calculated by the famous Clausius–Mossotti approximation (CMA)

$$\frac{\hat{\lambda}}{\lambda} = \frac{1 + 2\beta\nu}{1 - \beta\nu} \tag{5.53}$$

where $\beta = \frac{\lambda_1 - \lambda}{\lambda_1 + 2\lambda}$, ν is the concentration of the spheres. This formula is known also as the Maxwell-Garnett or Lorenz–Lorentz formula (for historical remarks see Refs. [40,41]). The formula (5.5.3) holds for dilute composites when the concentration ν is small.

In the 2D case, CMA becomes

$$\frac{\hat{\lambda}}{\lambda} = \frac{1 + \rho\nu}{1 - \rho\nu} \tag{5.54}$$

where $\rho = \frac{\lambda_1 - \lambda}{\lambda_1 + \lambda}$ is the 2D contrast parameter (see Eq. (5.22)). Here ν is the area concentration of disks on the plane (the section of the fiber composite perpendicular to the direction of fibers).

The Eqs. (5.53) and (5.54) can be deduced in the framework of Maxwell's formalism which is based on solution to the problem for one inclusion. The same method can be applied to inclusions of other shapes.

CMA in 2D Consider a disk of radius r_0 filled in by a material of conductivity λ_1. The disk $D = \{z \in \mathbb{C} : |z - a| < r_0\}$ is immersed in a material which occupies the domain $D_0 = \{z \in \mathbb{C} : |z - a| > r_0\}$ whose properties are described by the scalar conductivity λ. The whole material is placed into a constant macroscopic flux in the x_1-direction. Then, the following conjugation conditions hold on the circle $L = \{z \in \mathbb{C} : |z - a| = r_0\}$ orientated in the clockwise sense (see Eq. (5.10)):

$$T^+(t) = T^-(t), \quad \lambda \frac{\partial T^+}{\partial \mathbf{n}}(t) = \lambda_1 \frac{\partial T^-}{\partial \mathbf{n}}(t), \quad t \in L \tag{5.55}$$

We are looking for $T^{\pm}(z)$ in the following form:

$$T^-(r,\theta) = Ar\cos\theta + B, \quad T^+(r,\theta) = \frac{2\lambda_1}{\lambda_1 + \lambda}\left(r\cos\theta + \frac{C\cos\theta}{r} + E\right) \tag{5.56}$$

where $z = x_1 + ix_2 = a + r(\cos\theta + i\sin\theta)$, and A, B, C and E are undetermined constants. One of the constants B, E can be fixed arbitrarily since the temperature is uniquely determined up to an additive constant. It follows from Eq. (5.56) that the function $T^+(r,\theta)$ has the required behavior at infinity:

$$T^+(r,\theta) \sim \frac{2\lambda_1}{\lambda_1 + \lambda} r\cos\theta = \frac{2\lambda_1}{\lambda_1 + \lambda} x_1, \text{ as } r \to \infty$$

The conditions (5.55) in polar coordinates become

$$T^+(r_0,\theta) = T^-(r_0,\theta), \quad \lambda \frac{\partial T^+}{\partial r}(r_0,\theta) = \lambda_1 \frac{\partial T^-}{\partial r}(r_0,\theta), \quad 0 \le \theta < 2\pi \tag{5.57}$$

Substitute Eq. (5.56) in Eq. (5.57) gives

$$
\begin{aligned}
Ar_0\cos\theta + B &= \frac{2\lambda_1}{\lambda_1+\lambda}\left(r_0\cos\theta + \frac{C\cos\theta}{r_0} + E\right)\\
A\cos\theta &= \frac{2\lambda}{\lambda_1+\lambda}\left(1-\frac{C^2}{r_0}\right)\cos\theta,\quad 0\le\theta<2\pi
\end{aligned}
\tag{5.58}
$$

Taking $E = \mathrm{Re}\,a$, we obtain from Eq. (5.58):

$$
A = \frac{4\lambda\lambda_1}{(\lambda_1+\lambda)^2},\quad B = \frac{2\lambda_1}{\lambda_1+\lambda}\mathrm{Re}\,a,\quad C = \rho r_0^2 \tag{5.59}
$$

Therefore, the temperature distribution (5.56) has the form

$$
\begin{aligned}
T^-(r,\theta) &= \frac{2\lambda_1}{\lambda_1+\lambda}\left(\frac{2\lambda}{\lambda_1+\lambda}r\cos\theta + \mathrm{Re}\,a\right),\quad r\le r_0\\
T^+(r,\theta) &= \frac{2\lambda_1}{\lambda_1+\lambda}\left(r\cos\theta + \frac{\rho r_0^2\cos\theta}{r} + \mathrm{Re}\,a\right),\quad r\ge r_0
\end{aligned}
\tag{5.60}
$$

We introduce the complex potentials:

$$
\Phi^-(z) = T^-(z) + iV^-(z),\ z\in D,\ \Phi^+(z) = \frac{\lambda_1+\lambda}{2\lambda_1}(T^+(z)+iV^+(z)),\ z\in D_0 \tag{5.61}
$$

We write Eq. (5.55) in the form of the $\mathbb{R}$-linear conjugation problem (5.23):

$$
\Phi^+(t) = \Phi^-(t) - \rho\overline{\Phi^-(t)},\quad t\in L \tag{5.62}
$$

The function $\Phi^+(z)$ has the principal part $\frac{\lambda_1+\lambda}{2\lambda_1}z$ at infinity. It is easily checked that the following functions satisfy Eq. (5.62):

$$
\Phi^-(z) = (1-\rho^2)(z-a) + a - \rho\bar{a},\quad |z-a|\le r_0,\quad \Phi^+(z) = z + \frac{\rho r_0^2}{z-a},\quad |z-a|\ge r_0 \tag{5.63}
$$

The complex potentials (5.63) correspond to the harmonic functions (5.60) due to Eq. (5.61).

Following Maxwell's approach, we assume that ρr_0^2 is sufficiently small. This is true for dilute media when the inclusions are sparse or for weakly inhomogeneous media when the contrast parameter ρ is small. In the limit case $r_0 = 0$, the function $\Phi^-(z)$ degenerates to a constant and $\Phi^+(z) = z$. In the second limit case $\rho = 0$, Eq. (5.63) becomes

$$\Phi^-(z) = \Phi^+(z) = z \tag{5.64}$$

The case $r_0 = 0$ can be formally described also by Eq. (5.64). The difference between Eqs. (5.63) and (5.64) for $\Phi^+(z)$ is $\frac{\rho r_0^2}{z-a}$. The real part of the difference in polar coordinates becomes $\rho \frac{r_0^2 \cos\theta}{r}$. Hence, this term can be considered as a perturbation of the complex potential created by the external field due to one inclusion.

Consider the temperature distribution in the medium with n inclusions. The total perturbation term has the form $\rho n \frac{r_0^2 \cos\theta}{r}$ for sufficiently large r far away from all n inclusions. Take now a fictive "homogenized disk" of radius R_0 containing n small inclusions. Repeating the above argument, we will have a perturbation $\hat{\rho}\frac{R_0^2 \cos\theta}{r}$, where $\hat{\rho} = \frac{\hat{\lambda}-\lambda}{\hat{\lambda}+\lambda}$. Supposing that these perturbations are equal, we have

$$\rho n r_0^2 = R_0^2 \hat{\rho} \quad \Leftrightarrow \nu\rho = \frac{\hat{\lambda}-\lambda}{\hat{\lambda}+\lambda} \tag{5.65}$$

where $\nu = n(\frac{r_0}{R_0})^2$ is the area concentration of the disks. Solution to Eq. (5.65) yields CMA (5.54).

CMA in 3D CMA (5.54) can be found in the same way as in the previous part of this subsection. First, we are interested in the solution to the problem for one ball of radius r_0 centered at $\mathbf{x}_0 = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)})$:

$$T^+(\mathbf{t}) = T^-(\mathbf{t}), \quad \lambda \frac{\partial T^+}{\partial \mathbf{n}}(\mathbf{t}) = \lambda_1 \frac{\partial T^-}{\partial \mathbf{n}}(t), \quad |\mathbf{t}-\mathbf{x}_0| = r_0 \tag{5.66}$$

It can be directly checked that its solution has the form

$$T^-(\mathbf{x}) = \frac{3\lambda}{2\lambda+\lambda_1}(x_1 - x_1^{(0)}), \quad T^+(\mathbf{x}) = (x_1 - x_1^{(0)})\left(1 + \frac{\lambda-\lambda_1}{2\lambda+\lambda_1}\frac{r_0^3}{r^3}\right) \tag{5.67}$$

where $r = |\mathbf{x} - \mathbf{x}_0|$. Here, the external field along the x_1-axis is also considered.

In the limit cases $r_0 = 0$ and $\lambda_1 = \lambda$, the temperature distribution (5.67) becomes

$$T^+(\mathbf{x}) = T^-(\mathbf{x}) = x_1 - x_1^{(0)} \tag{5.68}$$

Therefore, one can consider the term $\frac{\lambda_1-\lambda}{2\lambda+\lambda_1}\frac{r_0^3}{r^3}$ $\beta\frac{r_0^3}{r^3}$ as a perturbation due to one ball. Then, the perturbation due to n balls $n\beta\frac{r_0^3}{r^3}$ can be equal to the perturbation of the homogenized ball:

$$\beta n r_0^3 = R_0^3 \frac{\hat{\lambda}-\lambda}{\hat{\lambda}+2\lambda} \Leftrightarrow \nu\beta = \frac{\hat{\lambda}-\lambda}{\hat{\lambda}+2\lambda} \tag{5.69}$$

where $\nu = n(\frac{r_0}{R_0})^3$ is the volume concentration of the balls. Solution to equation (5.69) yields CMA (5.53).

Remark 5.4.1 *Application of Maxwell's approach to composite media with elliptic inclusions is given in Ref. [42]. One can find there approximate analytical formulas for the effective conductivity. Other shapes of inclusions are studied in Ref. [43] in terms of Pòlya–Szege's matrix. This matrix expresses the effective conductivity tensor for dilute inclusions and corresponds to CMA.*

5.4.3 Effective Medium Theory (EMT)

The effective medium theory (EMT) is based on local study of the composites and porous media when an inclusion is embedded in a homogeneous medium whose effective conductivity is unknown and to be determined by averaging the local structure. Application of EMT leads to analytical formulas for the effective conductivity. It was originated by Bruggeman [44] and developed by Kirkpatric [45] who proposed approximating the medium by square or cubic network (see Section 5). Self-consistent methods, the method of Mori–Tanaka [46], differential effective medium methods and their relations and extensions were discussed in Ref. [47]. We discuss here the cases of circular and spherical inclusions separately.

EMT in 2D Consider a 2D basic element in the complex plane $\mathbb{C}$ in the form of a disk of radius r_k filled in by a material of conductivity λ_k, i.e. the number k is fixed. The disk $D_k = \{z \in \mathbb{C} : |z-a_k| < r_k\}$ is immersed in a material which occupies the domain $\Omega_k = \{z \in \mathbb{C} : |z-a_k| > r_k\}$ whose properties are described by the scalar conductivity $\hat{\lambda}$ (the effective conductivity to be determined later). The whole material is placed into a constant macroscopic flux in the x_1-direction. Then, the following conjugation conditions hold on the fixed circle $L_k = \{z \in \mathbb{C} : |z-a_k| = r_k\}$ orientated in the clockwise sense (compare with Eq. 5.55):

$$T^+(t) = T^-(t), \quad \hat{\lambda}\frac{\partial T^+}{\partial \mathbf{n}}(t) = \lambda_k \frac{\partial T^-}{\partial \mathbf{n}}(t), \quad t \in L_k \tag{5.70}$$

The problem (5.70) has been solved in Section 5.4.2 (see Eq. (5.60)). In polar coordinates $z = x_1 + ix_2 = a_k + r(\cos\theta + i\sin\theta)$, its solution has the form

$$T^-(r,\theta) = \frac{2\lambda_k}{\lambda_k+\lambda}\left(\frac{2\hat{\lambda}}{\lambda_k+\hat{\lambda}} r\cos\theta + \mathrm{Re}\, a_k\right)$$
$$T^+(r,\theta) = \frac{2\lambda_k}{\lambda_k+\lambda}\left(r\cos\theta + \frac{\Delta_k r_k^2\cos\theta}{t} + \mathrm{Re}\, a_k\right) \tag{5.71}$$

where $\Delta_k = \dfrac{\hat{\lambda}\lambda_k}{\hat{\lambda}+\lambda_k}$.

The problem (5.70) in terms of the complex potentials

$$\Phi^-(z) = T^-(z) + iV^-(z) z\in D_k,\;\; \Phi^+(z) = \frac{\lambda_k+\hat{\lambda}}{2\lambda_k}(T^+(z)+iV^+(z)),\;\; z\in\Omega_k \tag{5.72}$$

becomes

$$\Phi^+(t) = \Phi^-(t) - \Delta_k\overline{\Phi^-(t)}, \quad t\in L_k \tag{5.73}$$

It follows from Eq. 5.63 that

$$\Phi^-(z) = (1-\Delta_k^2)(z-a_k) + a_k - \Delta_k\overline{a_k}, \quad \Phi^+(z) = z + \frac{\Delta_k r_k^2}{z-a_k} \tag{5.74}$$

The residue of $\Phi^+(z)$ at $z = a_k$ is equal to $\Delta_k r_k^2$. It expresses the dipole moment of the complex potential $\Phi^+(z)$. Up to now it was assumed that the number k is fixed. Let us consider infinitely many disks packing the plane, i.e. each point of the plane belongs to one disk. Let all disks be divided into n types characterized by the radius r_k, and having conductivity λ_k. Let ν_k denote the concentration of the disks of the k-th type uniformly distributed on the plane. One can consider ν_k as the area probability of the k-th type. Then, $\sum_{k=1}^{n}\nu_k = 1$. We are interested in the total dipole moment corresponding to disks. The dipole moment of each disk is approximated by $\Delta_k r_k^2$ and the dipole moment of the ''level'' (phase) k is proportional to $\Delta_k\nu_k$. Supposing that the total moment created by each level vanishes, we have

$$\sum_{k=1}^{n}\nu_k\Delta_k = 0 \Leftrightarrow \sum_{k=1}^{n}\nu_k\frac{\hat{\lambda}-\lambda_k}{\hat{\lambda}+\lambda_k} = 0 \tag{5.75}$$

The effective conductivity $\hat{\lambda}$ is the solution of Eq. (5.75).

The relation (5.75) admits the following continuous interpretation:

$$\int_0^\infty \nu(\lambda)\frac{\hat{\lambda}-\lambda}{\hat{\lambda}+\lambda}\,d\lambda = 0 \tag{5.76}$$

where $\nu(\lambda)$ is the density function of the random variable λ.

Example 5.4.2 *Consider a two-phase material each phase of which takes the same area. Then,* $n = 2$, $\nu_1 = \nu_2 = \frac{1}{2}$ *and Eq. (5.75) yields the geometric mean for the effective conductivity*

$$\hat{\lambda} = \sqrt{\lambda_1 \lambda_2} \tag{5.77}$$

Example 5.4.3 *Following Ref. [47], consider the lognormal distribution of local conductivity*

$$\nu(\lambda) = \frac{1}{b\lambda\sqrt{\pi}} \exp\left(-\frac{(\ln\lambda - a)^2}{2b^2}\right) \tag{5.78}$$

with positive parameters a and b. We now prove that Eq. (5.76) with $\nu(\lambda)$ *given by Eq. (5.78) has the unique solution*

$$\hat{\lambda} = \exp a \tag{5.79}$$

The existence and uniqueness for Eq. (5.76) follows from physical observation that the effective conductivity exists for macroscopically isotropic media. Let us change the variable in the integral (5.76) $\lambda = \frac{\exp 2a}{t}$ *and use the relation* $\nu(t) = \frac{\exp 2a}{t^2} \nu\left(\frac{\exp 2a}{t}\right)$ *for the function (5.78). Then Eq. (5.76) becomes*

$$\int_0^\infty \nu(t) \frac{t - \dfrac{\exp 2a}{\hat{\lambda}}}{t + \dfrac{\exp 2a}{\hat{\lambda}}} \mathrm{d}t = 0 \tag{5.80}$$

Since Eqs. (5.76) and (5.80) have a unique solution, these equations give the same result $\hat{\lambda} = \frac{\exp 2a}{\hat{\lambda}}$ *which implies Eq. (5.79). It follows from Section 5.4.4 that the formula (5.80) is exact for the lognormal distribution.*

EMT in 3D Following the previous part of this subsection and Ref. [6], we construct an effective medium approximation in 3D case. It can be directly checked that the solution of the problem

$$T^+(\mathbf{t}) = T^-(\mathbf{t}), \quad \hat{\lambda} \frac{\partial T^+}{\partial \mathbf{n}}(\mathbf{t}) = \lambda_k \frac{\partial T^-}{\partial \mathbf{n}}(\mathbf{t}), \quad \mathbf{t} \in L_k \tag{5.81}$$

for the sphere L_k of radius r_k centered at $\mathbf{x}_k = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)})$ has the form

$$T^-(\mathbf{x}) = \frac{3\hat{\lambda}}{2\hat{\lambda} + \lambda_k}(x_1 - x_1^{(k)}), \quad T^+(\mathbf{x}) = (x_1 - x_1^{(k)})\left(1 + \frac{\hat{\lambda} - \lambda_k}{2\hat{\lambda} + \lambda_k} \frac{r_k^3}{r^3}\right) \tag{5.82}$$

where $r = |\mathbf{x} - \mathbf{x}_k|$. Here, for simplicity, the external field is taken along the x_1-axis.

The dipole moment corresponding to the phase of conductivity λ_k is proportional to $\nu_k \frac{\hat{\lambda}-\lambda_k}{2\hat{\lambda}+\lambda_k}$. Equating the total moment to zero, we obtain an equation for the effective conductivity $\hat{\lambda}$:

$$\sum_{k=1}^{n} \nu_k \frac{\widehat{\lambda}-\lambda_k}{2\widehat{\lambda}+\lambda_k} = 0 \tag{5.83}$$

The latter formula can be extended to the continuous phase case

$$\int_0^{\infty} \nu(\lambda) \frac{\hat{\lambda}-\lambda}{2\hat{\lambda}+\lambda} \mathrm{d}\lambda = 0 \tag{5.84}$$

Remark 5.4.4 *The differential method is closely related to EMT. One can find analytical formulas for different types of composite media and references for instance in Refs. [41,42].*

5.4.4
Duality Theory for 2D Media

Keller–Mathéron Theory The theory of duality transformation of 2D media was discovered by Keller [48] for two-phase media and independently by Mathéron [49] for general media. It is based on the observation that a divergent free field produces a curl-free field when it is rotated locally by 90°. Application of this theory yields an expression for the effective conductivity of a two-phase medium when phases are interchanged. The famous square-root formula (5.77) was deduced by Dykhne [50] by use of duality. It is exact for the square checkerboard.

Following Mathéron [49], we introduce the resistivity tensor $H(\mathbf{x}) = \Lambda^{-1}(\mathbf{x})$ and treat the local tensors as random functions of $\mathbf{x}$. Consider a 2D medium when the following hypothesis is fulfilled:

($\mathbf{H}_1$) *The random fields* $\Lambda(\mathbf{x})\langle\Lambda(\mathbf{x})\rangle^{-1}$ *and* $H(\mathbf{x})\langle H(\mathbf{x})\rangle^{-1}$ *have the same spatial distribution.*

Then [20,47,51,52],

$$|\hat{\Lambda}| = \frac{|\langle\Lambda(x)\rangle|}{|\langle\Lambda(x)|\Lambda(x)|^{-1}\rangle|} \tag{5.85}$$

where $|\cdot|$ denotes the determinant of the matrix, $\langle\cdot\rangle$ is the spatial average (5.42). In the particular case where $\Lambda(\mathbf{x})$ takes only two constant values $\lambda_1\mathbf{I}$ and $\lambda_2\mathbf{I}$ with the same probability $\frac{1}{2}$, we obtain Keller's identity [48]

$$\hat{\lambda}_1\hat{\lambda}_2 = \lambda_1\lambda_2 \tag{5.86}$$

Here, $\hat{\lambda}_1$ is the effective conductivity of the medium in the x_1-direction, $\hat{\lambda}_2$ is the effective conductivity in the x_2-direction of another medium which is produced from the original one by exchanging the phases with different conductivities λ_1 and λ_2.

Consider a medium for which the hypothesis ($\mathbf{H}_1$) and also the following hypothesis are valid:

($\mathbf{H}_2$) *The random fields $\Lambda(\mathbf{x})$ is invariant under rotations.*

Then, the effective tensor is isotropic and [49]

$$\hat{\lambda} = \sqrt{\lambda_0 h_0} \tag{5.87}$$

where the scalars λ_0 and h_0 denote the average conductivity and resistance over the representative cell, namely $\langle \Lambda(\mathbf{x}) \rangle = \lambda_0 \mathbf{I}$, $\langle H(\mathbf{x}) \rangle = h_0 \mathbf{I}$. In the particular case where $\Lambda(\mathbf{x})$ takes only two constant values $\lambda_1 \mathbf{I}$ and $\lambda_2 \mathbf{I}$ with the same probability $\frac{1}{2}$, Eq. (5.87) yields Eq. (5.77).

The following duality transformation between two different conductivity problems was proposed by Milton [3]. Let $\Lambda(\mathbf{x})$ be the conductivity tensor in a medium; define the conductivity tensor $\Lambda'(\mathbf{x})$ by

$$\Lambda'(\mathbf{x}) = [a\Lambda(\mathbf{x}) + b\mathbf{R}][c\mathbf{I} + d\mathbf{R}\Lambda(\mathbf{x})]^{-1} \tag{5.88}$$

where a, b, c, d are constants and $\mathbf{R}$ is the matrix for 90̂ rotation. Milton proved that their effective conductivity tensors are also related by Eq. (5.88).

The effective conductivity of the three-phase tesselation is conjectured to be an algebraic function in Ref. [53]. Other relations of the theory of duality transformations and their applications are discussed in Refs. [47,54–57] and works cited therein.

Craster–Obnosov Formulas It was noted in 1996 [58,59] that the study of the few-phases checkerboard composites can be reduced to the matrix Riemann–Hilbert problem of analytic function theory. Craster and Obnosov in a series the papers [59–65] deduced exact formulas for the effective conductivity tensor based on the explicit representations of the local fields for various types of such composites.

Consider a doubly periodic four-phase checkerboard composite when the representative rectangle has the lengths of the sides α, α^{-1} (see Figure 5.1).

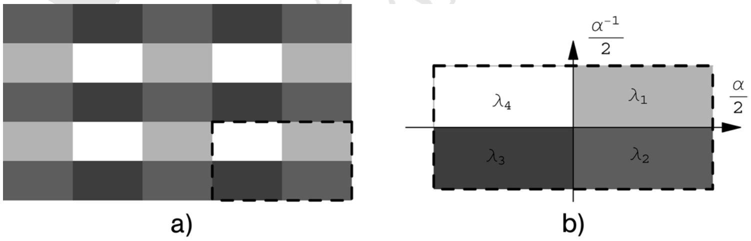

Fig. 5.1 Checkerboard four-phase medium: (a) doubly checkerboard; (b) representative cell.

Let the local conductivity $\lambda = \lambda(\mathbf{x})$ take the value λ_j in the j-phase ($j = 1,2,3,4$). Following Ref. [59], we use the complete elliptic integral:

$$K(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m^2 \sin^2\theta}}$$

where the parameter m is implicitly defined via the equation $K(m)/K(1-m) = \alpha^2$. We introduce also the parameters

$$\begin{aligned} &\sigma_1 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \quad \sigma_2 = \lambda_1\lambda_3 + \lambda_2\lambda_4, \\ &\sigma_3 = \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4. \end{aligned} \tag{5.89}$$

The parameter ϑ is introduced implicitly via $\cos\pi\vartheta = 1 - 2\sigma_2^2(\sigma_1\sigma_3 + \sigma_2^2)^{-1}$. We introduce the function

$$k(m, \vartheta) = \frac{P_{\frac{\vartheta}{2}-\frac{1}{2}}(2m-1)}{P_{\frac{\vartheta}{2}-\frac{1}{2}}(1-2m)} \tag{5.90}$$

where P_μ is the Legendre function of the first kind. The effective conductivity is explicitly given by the formulas

$$\hat{\lambda}_1 = \frac{1}{\alpha^2 k(m, \vartheta)} \left[\frac{(\lambda_1 + \lambda_2)(\lambda_3 + \lambda_4)}{(\lambda_2 + \lambda_3)(\lambda_4 + \lambda_1)}\right]^{\frac{1}{2}} \left(\frac{\sigma_1}{\sigma_3}\right)^{\frac{1}{2}} \tag{5.91}$$

$$\hat{\lambda}_2 = \alpha^2 k(m, \vartheta) \left[\frac{(\lambda_2 + \lambda_3)(\lambda_4 + \lambda_1)}{(\lambda_1 + \lambda_2)(\lambda_3 + \lambda_4)}\right]^{\frac{1}{2}} \left(\frac{\sigma_1}{\sigma_3}\right)^{\frac{1}{2}} \tag{5.92}$$

It is worth noting that for the square checkerboards $\alpha = 1$ implies $m = \frac{1}{2}$. Hence, $k\left(\frac{1}{2}, \vartheta\right) = 1$ and Eqs. (5.91) and (5.92) become simple algebraic functions of λ_j.

Other special cases such as $\lambda_2 = \lambda_4$, highly contrasting phases and other new formulas are discussed in the papers cited above. In particular, nonconducting or perfectly conducting strips between phases in four-phase checkerboards [61], triangle structures [62,64] have been explicitly studied. The case $\lambda_2 = \lambda_3 = \lambda_4$ corresponds to composites with rectangular inclusions of the concentration $\frac{1}{4}$. The effective properties of medium with quadratic holes with arbitrary concentration were explicitly calculated in Ref. [66] by application of conformal mappings.

5.5 Network Approximations

Network approximations for evaluation of the temperature field in composites and porous media is based on the replacement of the 3D solid material by Kirchhoff's discrete circuit presented by a graph. Such approximations can be applied for instance to two-phase composites with high contrast parameter when the conductivity of inclusions is much greater than the conductivity of matrix and the inclusions generate clusters, i.e. the inclusions are closed, and the heat is transferred mainly along paths connecting them. The vertices of the graph correspond to the inclusions and the paths correspond to one-dimensional edges of the graph.

Various authors [67–71] developed continuum percolation models taking into account the spatial geometry of media. The saddle points of the smooth phase function were used in Ref. [67] as elementary blocks of the network. By contrast, the capacity of the pair of inclusions were used in Refs. [68–71] as the conductivity of the basic paths conducting the heat.

Following Ref. [69], we discuss macroscopically isotropic 2D composites with random nonoverlapping identical circular inclusions of radii r. The temperature field between the neighboring disks D_i and D_j is described by Keller's formula [72] for the specific flux:

$$g_{ij} = \pi \sqrt{\frac{r}{\delta_{ij}}} \tag{5.93}$$

where δ_{ij} is the distance between the disks.

The definition of neighbor disks is introduced via 2D Voronoi tessellation. It is illustrated by Figure 5.2, where the centers of the disks are considered as vertices of the graph Γ. The domain Q is divided into polygonal regions. Each of the polygonal regions contains only one vertex and points which are closer to this vertex than to one another. Hence, each edge of the graph Γ is divided by a side of the polygons by two equal segments. Moreover, the edge and the side are perpendicular and all points of the side are equidistant from the corresponding vertices. The graph Γ' consisting of the sides of the polygons is called the dual graph to the graph Γ.

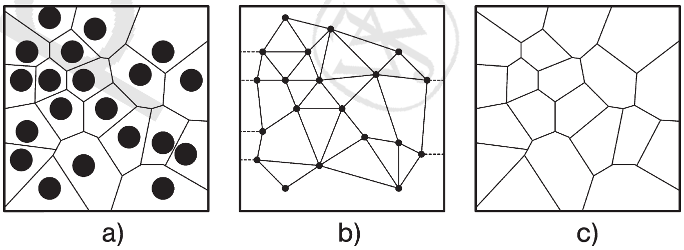

Fig. 5.2 Continuous percolation model: (a) nonoverlapping disks and Voronoi tessellation; (b) Delaunay graph Γ, its vertices are centers of the disks; (c) dual graph Γ' whose edges coincide with the sides of Voronoi's polygons.

Let i be the number of the disk D_i in the representative cell $Q(i=1,2,\ldots,N)$. Divide the set $\{i=1,2,\ldots,N\}$ onto the subsets I, S^+ and S^-, where $S^\pm$ corresponds to the disks which are closed to the boundary edges $x=\pm\frac{1}{2}$ of the cell Q and I corresponds to internal disks. The discrete temperature distribution $T(\mathbf{x})=t_i$, $\mathbf{x}\in D_i$, satisfies the system of linear algebraic equations:

$$\sum_{j\in A(i)} g_{ij}(t_i-t_j)=0, \quad \text{for } i\in I, \quad t_i=\pm\frac{1}{2}, \quad \text{for } i\in S^\pm \tag{5.94}$$

where g_{ij} has the form (5.93), $A(i)$ denotes the set of vertices which are adjacent to the vertex i, i.e. other vertices of the edges containing the vertex i. The first Eq. (5.94) correspond to the flux in the necks between inclusions, the second to the boundary conditions. Actually, Berlyand and co-workers, [68–71] used a more complicated model taking into account interaction with the boundary of Q.

The effective conductivity is given by the following asymptotic formula:

$$\hat{\lambda}=\frac{1}{4}\sum_{i,j} g_{ij}(t_i-t_j)^2+O(1)+O\left(\sqrt{\frac{\delta}{r}}\right), \quad \text{as } \delta\to 0 \tag{5.95}$$

where $\delta=\max(\delta_{ij})$. Each term $\frac{1}{4}g_{ij}(t_i-t_j)^2$ is of order $O\left(\sqrt{\frac{r}{\delta}}\right)$. It comes from the formula for the energy of a condenser $E=\frac{1}{2}CV^2$, where C is its capacity, V is the difference of potentials. The additional multiplier $\frac{1}{2}$ is put in Eq. (5.95), since each term is taken two times.

The formula (5.95) can be used for composites with a high concentration of inclusions and a high contrast parameter. First, one has to solve the system (5.94) with respect to t_i where the coefficients g_{ij} are determined by the distances between inclusions δ_{ij}, i.e. the geometry of the medium. Then, t_i are substituted into Eq. (5.95).

The correspondence of the discrete model to the original continuum medium with circular inclusions had been rigorously justified in Refs. [68–71]. Other shapes of inclusions and 3D composites were also discussed by a similar method. It follows from formula (5.93) that the capacity tends to infinity as δ_{ij} tends to zero. This means that the heat is mainly transferred through the necks between inclusions. However, the relationship between the discrete and continuous models is not direct for general problems. Kolpakov [73] investigated this problem by estimation of the capacity of the pairs of inclusions when the distance between the inclusions δ_{ij} tends to zero. He showed that the capacity tends to zero if $\delta_{ij}\to 0$ in 2D. However, this correspondence fails for the pair cone-half space in 3D.

In porous media, instead of the perfectly conducting inclusions, isolating holes are considered. Then, the conducting solid phase can be approximated by the graph Γ'. Pores can be filled by heated gases. Then, other types of problems can arise. It is worth noting that the latter problems are not studied as deeply as problems for porous media with perfectly conducting inclusions discussed above.

5.6
Doubly Periodic Problems

5.6.1
Introduction to Elliptic Function Theory

Elliptic functions [74,75] are doubly periodic meromorphic functions with the periods ω_1, ω_2 whose ratio ω_2/ω_1 is not a real number (it should be mentioned that Jacobi's theorem says that there is no single-valued analytic function with more than two periods).

Let us recall now some general properties of doubly periodic functions. Let $f(z)$ be a single-valued analytic function with two periods ω_1/ω_2, Im $\omega_2/\omega_1 > 0$. Then

$$f(z+w) = f(z), \quad \forall w = m_1\omega_1 + m_2\omega_2, \quad m_1, m_2 \in \mathbb{Z}$$

Any parallelogram with points z_0, $z_0 + \omega_1$, $z_0 + \omega_1 + \omega_2$, $z_0 + \omega_2$ as its vertices is called a *parallelogram of periods* (and also a *fundamental cell*, whenever $z_0 = 0$).

Weierstrass $\wp$-function The Weierstrass $\wp$-function can be represented in the form of a series:

$$\wp(z) = \frac{1}{z^2} + {\sum_{m_1,m_2}}' \left[\frac{1}{(z - m_1\omega_1 - m_2\omega_2)^2} - \frac{1}{(m_1\omega_1 + m_2\omega_2)^2}\right] \tag{5.96}$$

where ${\sum_{m_1,m_2}}'$ means that summation is performed over all integers $m_1, m_2 \in \mathbb{Z}$ except $m_1, m_2 = 0$. The properties (a)–(e) follow directly from Eq. (5.96):

a. $\wp(z)$ is a doubly periodic function with the only pole $m_1\omega_1 + m_2\omega_2$.
b. $\wp(z)$ is an even function of order 2.
c. Its derivative $\wp'$ is an odd function of order 3.
d. In the neighborhood of the origin its principal part is equal to $\frac{1}{z^2}$.
e. $\wp(z) - \frac{1}{z^2}$ tends to zero as $z \to 0$.
f. It is an inverse function to the elliptic integral:

$$z = \int_{\infty}^{\zeta} \frac{dx}{\sqrt{4x^3 - g_2 x - g_3}} (\zeta = \wp(z)) \tag{5.97}$$

where

$$g_2 = 60 {\sum_{m_1,m_2}}' \frac{1}{(m_1\omega_1 + m_2\omega_2)^4}, \quad g_3 = 140 {\sum_{m_1,m_2}}' \frac{1}{(m_1\omega_1 + m_2\omega_2)^6} \tag{5.98}$$

The functions $\wp(z)$ and $\wp'$ are related by the following algebraic relation:

$$\wp'^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3 \tag{5.99}$$

Weierstrass ζ-function The Weierstrass ζ-function (which should not be confused with the well-known Riemann ζ-function playing an important role in the description of prime numbers) is defined by integration of the Weierstrass ℘-function, namely

$$\zeta(z) = \frac{1}{z} - \int_0^z \left\{ \wp(z) - \frac{1}{z^2} \right\} dz \tag{5.100}$$

Differentiating Eq. (5.100), we have $\zeta'(z) = -\wp(z)$. Therefore, the Weierstrass ζ-function is an odd function ($\zeta(-z) = -\zeta(z)$) having only the one pole in the parallelogram of periods (of the ℘-function (!)) with residue 1. Hence, the ζ-function cannot be elliptic; sometimes it is called *quasi-periodic* since the following relations are valid:

$$\zeta(z + \omega_1) = \zeta(z) + 2\eta_1, \quad \zeta(z + \omega_2) = \zeta(z) + 2\eta_2 \tag{5.101}$$

where the constants η_1, η_2 are equal $\eta_1 = \zeta(\omega_1/2)$, $\eta_2 = \zeta(\omega_2/2)$.

Weierstrass σ-function This function of the Weierstrass collection is obtained due to line integration of $\zeta(z) - \frac{1}{z}$ along an arbitrary curve starting from the origin and not passing through any pole of the integrand. To avoid multi-valuedness, $\sigma(z)$ is defined as follows:

$$\log \frac{\sigma(z)}{z} = \int_0^z \left\{ \zeta(z) - \frac{1}{z} \right\} dz \tag{5.102}$$

Differentiation formula has in this case the form $\frac{\sigma'(z)}{\sigma(z)} = \zeta(z)$. The σ-function is an odd function having no singularity in any bounded domain and only zeroes at $z = m_1\omega_1 + m_2\omega_2$. Thus, it is also not an elliptic function. The following holds with the same constants as for ζ-function: $\sigma(z + \omega_1) = -e^{2\eta_1(z+\omega_1)}\sigma(z)$, $\sigma(z + \omega_2) = -e^{2\eta_2(z+\omega_2)}\sigma(z)$.

θ-function In practice, it is often supposed that one of the periods of an elliptic function is real. It can be realized by performing the following changes of variables: $v = \frac{z}{\omega_1}$, $\tau = \frac{\omega_2}{\omega_1}$. Then, in the variable ν, the elliptic function will have the periods 1 and τ, still supposing that Im $\tau > 0$. In this variable, the θ-function is defined in form of a series:

$$\theta(v) = i \sum_{-\infty}^{\infty} (-1)^n q^{\left(n-\frac{1}{2}\right)^2} e^{(2n-1)\pi v i} \tag{5.103}$$

where $q = e^{\pi i \tau}$. There is a connection between the θ-function and σ-function given by the formula:

$$\sigma(z) = \frac{\omega_1}{\theta'(0)} e^{\frac{\pi z^2}{\omega_1}} \theta\left(\frac{z}{\omega_1}\right) \tag{5.104}$$

Hence, the θ-function has no poles at any bounded domain (and is not an elliptic function). From the definition (5.103) of the θ-function, it follows that $\theta(v+1) = -\theta(v)$, $\theta(v+\tau) = -\frac{1}{q}e^{-2\pi v i}\theta(v)$. The points $v = m_1 + m_2\tau$ are the only zeros of the θ-function.

Classical Eisenstein–Rayleigh Sums It is convenient to use the elliptic functions in the form of the *Eisenstein series* introduced by Eisenstein in 1847 and developed by Weil [76]. The classical lattice sums (the Eisenstein sums) were applied to calculation of the effective conductivity tensor by Rayleigh [77] (see also Refs. [78,79]) when a representative cell contains one inclusion.

In the present subsection, we introduce the fundamental parameters of the elliptic function theory following Weil [76] and Akhiezer [74]. Consider a lattice $\mathcal{Q}$ which is defined by two fundamental translation vectors expressed by complex numbers ω_1 and ω_2 on the complex plane $\mathbb{C}$. For definiteness, we assume that $\mathrm{Im}\,\tau > 0$, where $\tau = \omega_{2/}\omega_1$. We introduce the $(0,0)$ cell $Q_{(0,0)}: = \{z = t_1\omega_1 + t_2\omega_2: -1/2 < t_j < 1/2 (j = 1,2)\}$. The lattice $\mathcal{Q}$ consists of the cells $Q_{(m_1,m_2)}: \{z \in \mathbb{C} : z - m_1\omega_1 - m_2\omega_2 \in Q_{(0,0)}\}$, where m_1 and m_2 run over integer numbers.

The Eisenstein summation method is defined as follows:

$$\sum_{m_1,m_2} = \lim_{N\to\infty} \sum_{m_2=-N}^{N} \left(\lim_{M\to\infty} \sum_{m_1=-M}^{M} \right) \tag{5.105}$$

Using this summation, we introduce

$$S_n(\omega_1, \omega_2): \sum_{m_1,m_2}{}' (m_1\omega_1 + m_2\omega_2)^{-n} \tag{5.106}$$

where m_1 and m_2 run over all integer numbers except the pair $m_1 = m_2 = 0$, $n = 2,3,\ldots$. The sum (5.106) with $n = 2$ is conditionally hence slowly convergent. The formula deduced in Ref. [80] $S_2(\omega_1, \omega_2) = \frac{2}{\omega_1}\zeta\left(\frac{\omega_1}{2}\right)$ is efficient in computations. Rylko [91] deduced another efficient formula:

$$S_2(\omega_1, \omega_2) = \left(\frac{\pi}{\omega_1}\right)^2 \left(\frac{1}{3} - 8\sum_{m=1}^{\infty} \frac{mq^{2m}}{1-q^{2m}}\right), \text{ where } q = \exp(\pi i \tau) \tag{5.107}$$

The sums (5.106) with $n > 2$ are absolutely convergent. It is known that $S_n(\omega_1,\omega_2) = 0$ for odd n. For even n, the Eisenstein–Rayleigh sums (5.106) can be easily calculated through the rapidly convergent infinite sums (see Eq. (5.98)

$$g_2 = g_2(\omega_1, \omega_2) = \left(\frac{\pi}{\omega_1}\right)^4 \left(\frac{4}{3} + 320 \sum_{m=1}^{\infty} \frac{m^3 q^{2m}}{1-q^{2m}}\right) \tag{5.108}$$

$$g_3 = g_3(\omega_1, \omega_2) = \left(\frac{\pi}{\omega_1}\right)^6 \left(\frac{8}{27} - \frac{448}{3} \sum_{m=1}^{\infty} \frac{m^5 q^{2m}}{1-q^{2m}}\right) \tag{5.109}$$

Then, $S_4(\omega_1, \omega_2) = \frac{1}{60} g_2(\omega_1, \omega_2)$, $S_6(\omega_1, \omega_2) = \frac{1}{1400} g_3(\omega_1, \omega_2)$. The sums $S_{2n}(\omega_1,\omega_2)$ $(n \geq 4)$ are calculated by the recurrence formula:

$$S_{2n}(\omega_1, \omega_2) = \frac{3}{(2n+1)(2n-1)(n-3)} \sum_{m=2}^{n-2} (2m-1)(2n-2m-1) S_{2m} S_{2(n-m)} \tag{5.110}$$

Remark 5.6.1 *The series (5.106) with $n = 2$ is Ref. [82] conditionally convergent. The possibility of its calculation by using (5.107) is justified in. A formula for nonperiodic (random) arrays to properly define S_2 is discussed in Ref. [83].*

Weierstrass functions can be expressed as Taylor expansions:

$$\ln \sigma(z) = \ln z - \sum_{n=2}^{\infty} \frac{S_{2n}}{2n} z^{2n}, \quad \zeta(z) = \frac{1}{z} - \sum_{n=2}^{\infty} S_{2n} z^{2n-1},$$
$$\wp(z) = \frac{1}{z^2} + \sum_{n=2}^{\infty} (2n-1) S_{2n} z^{2n-2} \tag{5.111}$$

The formulas (5.111) are not used for calculating the Weierstrass functions. For instance, $\sigma(z)$ is better computed by (5.104), because the θ–function can be computed by a very fast formula (5.103).

Eisenstein Series In the following, we summarize the main facts of the Eisenstein series theory following Weil [76]. The Eisenstein series are defined as follows:

$$E_n(z; \omega_1, \omega_2): \sum_{m_1, m_2} (z - m_1\omega_1 - m_2\omega_2)^{-n}, \quad n = 2, 3, \ldots \tag{5.112}$$

The Eisenstein summation method (5.105) is applied to $E_2(z;\ \omega_1,\omega_2)$. The series $E_n(z;\ \omega_1,\omega_2)$ for $n = 3,\ 4,\ \ldots$ as a function in z converge absolutely and almost uniformly in the domain $\mathbb{C} \setminus \cup_{m_1,m_2} (m_1\omega_1 + m_2\omega_2)$. Each of the functions (5.112) is doubly periodic and has a pole of order n at $z = 0$. However, further it will be convenient to define the value of $E_n(z;\ \omega_1,\omega_2)$ at the point zero as follows:

$$E_n(0; \omega_1, \omega_2): S_n(\omega_1, \omega_2) \tag{5.113}$$

The Eisenstein functions of the even order $E_{2n}(z)$ can be presented in the form of the series:

$$E_{2n}(z) = \frac{1}{z^{2n}} + \sum_{k=0}^{\infty} \sigma_k^{(n)} z^{2(k-1)} \tag{5.114}$$

where

$$\sigma_k^{(n)} = \frac{(2n+2k-3)!}{(2n-1)!(2k-2)!} S_{2(n+k-1)} \tag{5.115}$$

The Eisenstein series and the Weierstrass function $\wp(z;\omega_1,\omega_2)$ are related by the identities

$$\begin{aligned} E_2(z;\omega_1,\omega_2) &= \wp(z;\omega_1,\omega_2) + S_2(\omega_1,\omega_2), \\ E_n(z;\omega_1,\omega_2) &= \frac{(-1)^n}{(n-1)!}\frac{d^{n-2}}{dz^{n-2}}\wp(z;\omega_1,\omega_2) \end{aligned} \tag{5.116}$$

Generalized Eisenstein–Rayleigh Sums We now proceed to introduce one of the most important mathematical objects of the present section, the generalized Eisenstein–Rayleigh sums. Consider a set of points a_k ($k = 1, 2, \dots, N$) in the cell Q. Let p be a natural number; k_s runs over 1 to N, $n_j = 2, 3, \dots$. Let $\mathbf{C}$ be the operator of complex conjugation. The value

$$e_{m_1\dots m_q} := N^{-[1+\frac{1}{2}(m_1+\dots+m_q)]} \sum_{k_0 k_1 \dots k_q} E_{m_1}(a_{k_0}-a_{k_1})\overline{E_{m_2}(a_{k_1}-a_{k_2})}\dots\mathbf{C}^q E_{m_q}(a_{k_{q-1}}-a_{k_q}) \tag{5.117}$$

is called the generalized Eisenstein–Rayleigh sum. The parameters ω_1 and ω_2 are omitted in E_n. According to (5.113), $e_n(\omega_1, \omega_2)$ becomes the classical Eisenstein–Rayleigh sum $S_n(\omega_1, \omega_2)$ in the case $N = 1$.

We are also interested in the normalized Eisenstein series (compare to Eq. (5.103):

$$E_n(z;1,\tau) : \sum_{m_1,m_2} (z-m_1-m_2\tau)^{-n}, \; n = 2, 3, \dots \tag{5.118}$$

We have the relations

$$E_n(z;\omega_1,\omega_2) = \omega_1^{-n} E_n\left(\frac{z}{\omega_1};1,\tau\right), \; e_{n_1\dots n_p}(\omega_1,\omega_2) = \omega_1^{-2k} e_{n_1\dots n_p}(1,\tau) \tag{5.119}$$

where $2k := n_1 + \ldots + n_p$. Note that we shall need further only the even sums $n_1 + \ldots + n_p$.

Remark 5.6.2 *Berdichevskij [84] constructed three-dimensional counterparts of the elliptic functions which could be used for three-dimensional conductivity and elasticity problems. Huang [85] proposed exact integral formulas for three-dimensional lattice sums. His examples show that simple quadrature rules with modest numbers of nodes yield highly accurate results. A review of various numerical calculations of three-dimensional lattice sums is given in Ref. [85].*

5.6.2
Method of Functional Equations

In this section, we describe the method of functional equations in the class of analytical functions [16,80,86–90]. This method is used for solution of boundary value problems for the Laplace equation.

Let us consider the method of functional equations in the example of a solution of the following problem. Consider the cell $Q_{(0,0)}$ with N nonoverlapping circular disks D_k of radius r with the centers $a_k \in Q_{(0,0)} (k = 1,2,\ldots, N)$. Let D_0 be the complement of the closure of all disks D_k to $Q_{(0,0)}$. We study the conductivity of the doubly periodic composite material when the domains $D_{per} := \cup_{(m_1,m_2)}(D_0 \cup \partial Q_{(0,0)} + m_1\omega_1 + m_2\omega_2)$ and $D_k + m_1\omega_1 + m_2\omega_2$ (m_1, m_2 are integers) are occupied by materials of conductivities λ_0 and λ, respectively. The conductivity of the inclusions λ is expressed relative to λ_0. Hence, the conductivity of the matrix can be taken as unity ($\lambda_0 = 1$). The local potential $T(z)$ in $Q_{(0,0)}$ satisfies the conjugation conditions:

$$T^+(t) = T^-(t), \frac{\partial T^+}{\partial n}(t) = \lambda \frac{\partial T^-}{\partial n}(t) \text{ on } \partial D_k = \{t \in \mathbb{C} : |t - a_k| = r\},$$
$$k = 1, 2, \ldots, N \tag{5.120}$$

The potential $T(z)$ satisfies the quasi-periodicity conditions:

$$T(z + \omega_1) = T(z) + \Omega_1, \quad T(z + \omega_2) = T(z) + \Omega_2 \tag{5.121}$$

Here, the function $T(z)$ is harmonic in $Q_{(0,0)}$ except $\partial D_k (k = 1, 2, \ldots, N)$, the circles ∂D_k are orientated in the clockwise direction. In order to determine the effective conductivity tensor Λ, it is sufficiently to solve the problem (5.120), (5.121) with two linear independent vectors (Ω_1, Ω_2). The problem (5.120) is reduced to the $\mathbb{R}$-linear conjugation problem (see Section 5.2.4). For simplicity, consider the case $N = 1$:

$$\psi(t) = \psi_1(t) + \rho\left(\frac{r}{t-a}\right)^2 \overline{\psi_1(t)} - 1, \; |t-a| = r \tag{5.122}$$

The unknown function $\psi(z)$ can be presented in the form of its Taylor expansion: $\psi(z) = \rho\sum_{l=0}^{\infty}\psi_l(z-a)^l$. The problem (5.122) is reduced to the following functional equation:

$$\psi(z) = \rho \sum_{l=0}^{\infty} \overline{\psi_l} r^{2(l+1)} \{E_{l+2}(z-a) - (z-a)^{-(l+2)}\} + 1, \quad |z-a| \le r \tag{5.123}$$

We look for $\psi(z)$ in the form of the series expansion in $r^2 : \psi(z) = \sum_{s=0}^{\infty} r^{2s} \psi^{(s)}(z)$. The functional Eq. (5.123) has a unique solution which can be found by the method of successive approximation uniformly convergent in $|z-a| \le r$:

$$\begin{aligned} &\psi^{(0)}(z) = 1, \\ &\psi^{(p+1)}(z) = \rho[\overline{\psi_p^{(0)}} \eta_{p+2}(z-a) + \overline{\psi_{p-1}^{(1)}} \eta_{p+1}(z-a) + \ldots + \overline{\psi_0^{(p)}} \eta_2(z-a)] \end{aligned} \tag{5.124}$$

where $\eta_p(z) = E_p(z) - z^{-p}$. Using Eqs. (5.50) and (5.124), Rylko [81] calculated approximately $\hat{\lambda}$ for a square array of cylinders and $\rho = 1$:

$$\hat{\lambda} = \frac{1+\nu}{1-\nu} + 6 S_4^2 \pi^{-4} \frac{\nu^5}{(1-\nu)^2} + 2(9 S_4^2 + 7 S_8^2) \pi^{-8} \nu^9 + O(\nu^{10}) \tag{5.125}$$

where $\nu = \frac{N \pi r^2}{|Q_{(0,0)}|}$ is the concentration of the disks in the cell $Q_{(0,0)}$, $|Q_{(0,0)}|$ is the area of $Q_{(0,0)}$, $S_2 = \pi$, $S_4 \approx 3.1512112$, $S_8 \approx 4.2557732$. The first term in Eq. (5.125) corresponds to the CMA (5.54). More complicated investigation of the functional Eq. (5.123) implies the following exact formula for a square array:

$$\begin{aligned} \hat{\lambda} = {} & 1 + 2\rho\nu + 2\rho^2\nu^2 \\ & + 2\rho^2\nu^2 \frac{1}{\pi} \sum_{k=1}^{\infty} \rho^k \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \ldots \sum_{m_k=1}^{\infty} \sigma_{m_1}^{(1)} \sigma_{m_2}^{(m_1)} \ldots \sigma_{m_k}^{(m_{k-1})} \sigma_1^{(m_k)} \left(\frac{\nu}{\pi}\right)^{2(m_1+m_2+\ldots+m_k)-k} \end{aligned} \tag{5.126}$$

where $\sigma_k^{(n)}$ has the form (5.115).

Similar arguments can be applied to the general $\mathbb{R}$-linear conjugation problem with arbitrary N. The effective conductivity tensor $\hat{\Lambda}$ has the following structure:

$$\hat{\Lambda} = (1 + 2\rho\nu) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2\rho\nu \sum_{k=1}^{\infty} \begin{pmatrix} \mathrm{Re} A_k & \mathrm{Im} A_k \\ \mathrm{Im} A_k & C_k \end{pmatrix} \nu^k \tag{5.127}$$

where

$$A_k = |Q_{(0,0)}|^k \sum_{n_1 \ldots n_p} B_{n_1 \ldots n_p}^{(k)} e_{n_1 \ldots n_p}(\omega_1, \omega_2) \tag{5.128}$$

The constants $B^{(k)}_{n_1 \ldots n_p}$ depend only on k, ρ and $n_1, \ldots, n_p$. Here, $n_j = 2, 3, \ldots$; $k = 1, 2, \ldots$. The values C_k have an analogous form. Only the terms $e_{n_1 \ldots n_p}(\omega_1, \omega_2)$ defined by (5.117) depend on the centers of inclusions a_k in the representation (5.127) of $\hat{\Lambda}$. The first few coefficients A_k have the form [98]

$$\begin{aligned}
&A_1 = \frac{\rho}{\pi} e_2, \quad A_2 = \frac{\rho^2}{\pi^2} e_{22},\ A_3 = \frac{1}{\pi^3}[-2\rho^2 e_{33} + \rho^3 e_{222}] \\
&A_4 = \frac{1}{\pi^4}[3\rho^2 e_{44} - 2\rho^3(e_{332} + e_{233}) + \rho^4 e_{2222}] \\
&A_5 = \frac{1}{\pi^5}[-4\rho^2 e_{55} + \rho^3(3e_{442} + 6e_{343} + 3e_{244}) \\
&\quad -2\rho^4(e_{3322} + e_{2332} + e_{2233}) + \rho^5 e_{22222}] \\
&A_6 = \frac{1}{\pi^6}[5\rho^2 e_{66} - 4\rho^3(e_{255} + 3e_{354} + 3e_{453} + e_{552}) \\
&\quad + \rho^4(3e_{2244} + 6e_{2343} + 4e_{3333} + 3e_{2442} + 6e_{3432} + 3e_{4422}) \\
&\quad -2\rho^5(e_{22233} + e_{22332} + e_{23322} + e_{33222} + \rho^6 e_{222222})]
\end{aligned} \tag{5.129}$$

where the argument (ω_1, ω_2) is omitted. In particular, for macroscopically isotropic composites (5.127) becomes

$$\hat{\lambda} = 1 + 2\rho\nu + 2\rho\nu \sum_{k=1}^{\infty} A_k \nu^k \tag{5.130}$$

5.7 Representative Cell

One of the most important notation of composites and porous media is the *representative volume element* (RVE) or *representative cell* already used in this chapter. One can give a vague physical definition of this term as follows. RVE is a part of material which is small enough from a macroscopic point of view that it can be treated as a typical element of the heterogeneous medium. On the other hand, it is sufficiently large in the microscopic scale that it represents a typical microstructure of the material under consideration. In the present section following Ref. [91], we first give a rigorous definition of the representative element and then determine its minimal size. The geometrical interpretation of the problem is shown in Figure 5.3. The large cell Q' presented in Figure 5.3a is replaced by a smaller one, Q (see Figure 5.3b) with three inclusions per periodic cell.

Note that Adler and co-workers [5,6] discussed questions of the reconstruction of porous media by statistical data and a numerically constructed RVE.

Consider a two-dimensional two-component periodic composite medium made from a collection of nonoverlapping identical circular disks embedded in an otherwise uniform matrix. Let the inclusions have scalar conductivity λ and be separated by a matrix of unit conductivity. Let $\rho = (\lambda - 1)/(\lambda + 1)$ be the contrast parameter. It is established in Section 5.6.2 that the effective conductivity tensor $\hat{\Lambda}$ has the form of a

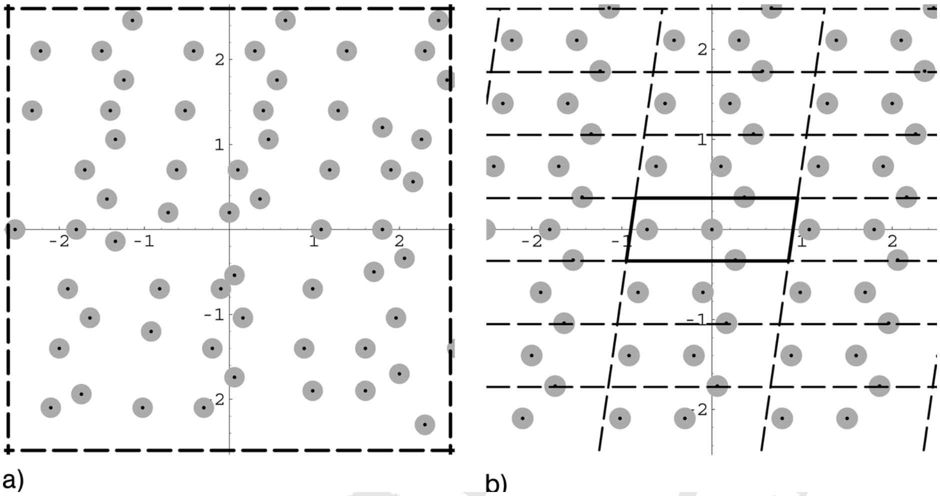

Fig. 5.3 Representative cells.

double series in the concentration of inclusions and on "basic elements" which depend only on locations of the inclusions (see Eqs. (5.127) and (5.128). These basic elements are written in terms of the Eisenstein series. Coefficients in the double series depend on ρ. We say that two composites are equivalent if expansions of their $\hat{\Lambda}$ have the same basic elements. Therefore, we divide the set of the composites with circular identical inclusions into classes of equivalence determined only by geometrical structure of the composite. In particular, composites with the same locations of inclusions but with different ρ belong to the same class of equivalence. Note that composites belonging to a class of equivalence can have different $\hat{\Lambda}$; and composites from different classes can have the same $\hat{\Lambda}$. Each composite material is represented by a periodic cell. In each class of equivalence, we choose a composite having the minimal size cell. This cell is called *the representative cell* of the considered class of equivalent composite materials.

We propose a constructive algorithm to determine the representative cell for any distribution of inclusions using only pure geometrical parameters. More precisely, at the beginning, we calculate the generalized Eisenstein–Rayleigh sums (5.117) depending on the centers of circular inclusions for given large cell. Then using these sums, we construct the (minimal) representative cell, i.e. we calculate its fundamental translation vectors and determine the positions of inclusions within this cell.

Consider a large fundamental region Q' constructed by the fundamental translation vectors ω_1' and ω_2'. Let Q' contain N' non-overlapping circular disks D_k' of radius r with the centers $a_k' \in Q$ ($k = 1, 2, \ldots, N'$). Let $\hat{\Lambda}'$ be the effective conductivity tensor of the composite material represented by the region Q' with inclusions D_k'. We are interested in the following question. To replace Q' by another small cell Q which contains inclusions $D_k = \{z \in \mathbb{C} : z - a_k| < |r\}$ ($k = 1, 2, \ldots, N$) and which has an effective conductivity tensor $\hat{\Lambda}$ close to $\hat{\Lambda}'$. We assume that the concentration ν of the inclusions in both materials is the same. Closeness is defined by the accuracy $O(\nu^{L+1})$ for the difference $\Delta\hat{\Lambda} = \hat{\Lambda} - \hat{\Lambda}'$ with prescribed L. We say that Q is a representative cell for the region Q' with the accuracy $O(\nu^{L+1})$ if $\Delta\hat{\Lambda} = O(\nu^{L+1})$. We say that Q is the minimal representative cell for the region Q' if Q is a representative cell with minimal possible area $|Q|$. For brevity, we further call the minimal representative cell the representative cell. The existence of the representative cell is evident since in the worst case one can take $Q = Q'$.

We adopt the designations generalized Eisenstein–Rayleigh sums for the representative cell. Consider Eq. (5.127) for the large cell Q'

$$\hat{\Lambda}' = (1 + 2\rho\nu)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2\rho\nu \sum_{k=1}^{\infty} \begin{pmatrix} \mathrm{Re}A'_k & \mathrm{Im}A'_k \\ \mathrm{Im}A'_k & C'_k \end{pmatrix} \nu^k \quad (5.131)$$

$$A'_k = |Q'|^k \sum_{n_1 \ldots n_p} B^{(k)}_{n_1 \ldots n_p} e_{n_1 \ldots n_p}(\omega'_1, \omega'_2) \quad (5.132)$$

Note that the coefficients $B^{(k)}_{n_1 \ldots n_p}$ have the same form in Eqs. (5.128) and (5.131). $\Delta\hat{\Lambda}$ is of order $O(\nu^{L+1})$ if $A'_k = A_k$ for $k = 1, 2, \ldots, L-1$. Therefore, $\Delta\hat{\Lambda}$ is of order $O(\nu^{L+1})$ if and only if

$$|Q|^k e_{n_1 \ldots n_p}(\omega_1, \omega_2) = |Q'|^k e_{n_1 \ldots n_p}(\omega'_1, \omega'_2) \quad (5.133)$$

for $k = 1, 2, \ldots, L-1$ and corresponding sets of the numbers $n_1, \ldots, n_p$. According to our definition, Q is a representative cell for the region Q' with the accuracy $O(\nu^{L+1})$ if and only if the relations Eq. (5.132) are fulfilled.

One can consider Eq. (5.132) as a system of equations with respect to $\omega_1, \omega_2, a_1, a_2, \ldots, a_N$ including the unknown number N with the restriction $|a_j - a_m| \leq 2r$ $(j \neq m)$. One can assume that one of the centers, say a_N, lies at the origin, since geometrically any cell is determined up to translation. The fundamental region Q as well as the translation vectors ω_1, ω_2 can be chosen in infinitely many ways [74]. For any doubly periodic structure on the plane, it is always possible to construct such a pair ω_1, ω_2 that $\omega_1 > 0$ and $\mathrm{Im}\tau > 0$.

The area of Q is calculated by ω_1 and ω_2

$$|Q| = \omega_1^2 \mathrm{Im}\tau \quad (5.134)$$

On the other hand, we also have $|Q| = \dfrac{N\pi r^2}{\nu}$ that yields the formula

$$\omega_1 = \sqrt{\frac{N\pi r^2}{\nu \mathrm{Im}\tau}} \quad (5.135)$$

In order to construct the representative cell with the prescribed accuracy $O(\nu^{L+1})$, we propose to solve the system (5.132) with fixed L increasing the number of inclusions in the cell N from 1 to N'. Then, N is fixed in each step of the study of Eq. (5.132).

Applying Eqs. (5.119) and (5.133), we rewrite Eq. (5.132) in the form

$$(\mathrm{Im}\tau)^k e_{n,\ldots,n_p}(1,\tau) = |Q'|^k e_{n_1,\ldots,n_p}(\omega_1', \omega_2'), \quad k = 1, 2, \ldots, L-1 \tag{5.136}$$

We can consider Eq. (5.135) as a system with respect to τ, a_1, a_2, ..., a_{N-1} ($a_N = 0$) with the restriction $|a_j - a_m| \geq 2r$ ($j \neq m$). The right-hand part of Eq. (5.135) is known. If we know a solution of Eq. (5.135), we can calculate ω_1 from Eq. (5.134).

It is also possible to state the problem of the representative cell with prescribed form of the cell Q. Let us consider the case when Q is a rectangle. Then, $\tau = i\alpha$, where α is positive and Eq. (5.134) implies that

$$\omega_1 = \sqrt{\frac{N\pi r^2}{\alpha\nu}} \tag{5.137}$$

Equations (5.135) become

$$\alpha^k e_{n_1,\ldots,n_p}(1, i\alpha) = |Q'|^k e_{n_1,\ldots,n_p}(\omega_1', \omega_2'), \quad k = 1, 2, \ldots, L-1 \tag{5.138}$$

Numerical examples of solution to Eqs. (5.135), (5.137) are presented in Ref. [91]. One can also find there a discussion devoted to other shapes of inclusions.

A spatial theory of the representative elements can also be constructed due to Berdichevskij 3D analogs of the elliptic functions (see Remark 5.6.2).

5.8 Nonlinear Heat Conduction

In the general mathematical theory of the nonlinear behavior of materials, the conductivity (for instance, electric) depends locally on the gradient of the potential. However, in the thermal conductivity, we have another dependence. Namely, the local coefficient depends on the potential, i.e. on the temperature distribution. Then, Fourier's law (5.1) becomes

$$\mathbf{q} = -\lambda(T)\nabla T \tag{5.139}$$

This case is easier to study due to the transformation (5.1.3) from Ref. [16 Chapter V].

There is a wonderful result in the nonlinear homogenization theory due to Artola and Duvaut [92] (see also Ref. [93]) devoted to homogenization of Eq. (5.138). It follows from Ref. [92] that in order to obtain a formula for the effective conductivity, it is sufficient to take a formula from linear theory with constant λ and to substitute $\lambda(T)$ instead of this constant. For instance, in the linear case, $\hat{\Lambda}$ for laminates is determined by Eqs. (5.44) and (5.52). Let the conductivities of the constituents depend on T, i.e. $\lambda_j = \lambda_j(T)$ ($j = 1, 2$). Then, the effective conductivities in this nonlinear case are exactly calculated by the same formulas (5.44) and (5.52) but with replacing $\hat{\lambda}_j$ and λ_j by the functions $\hat{\lambda}_j(T)$ and $\lambda_j(T)$.

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References

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1 Allair, G. (2002). *Shape Optimization by the Homogenization Method*, Springer Verlag.
2 Cherkaev, A. (2000) *Variational Methods for Structural Optimization*, Springer Verlag.
3 Milton, G.W. (2002) *The Theory of Composites*, Cambridge University Press.
4 Wojnar, R., Bytner, R. and Galka, A. (1999) Effective properties of elastic composites subject to thermal fields. in *Thermal Stresses V* (ed. R.B. Hetnarski), Lastran Corp., Publ. Division Rochester, pp.257–465.
5 Adler, P.M. (1992) *Porous Media. Geometry and Transport*, Butterworth-Heinemann.
6 Adler, P.M. and Thovert, J.-F. (1999) *Fractures and Fracture Networks*, Butterworth-Heinemann.
7 de Boer, R. (2005) *Trends in Continuum Mechanics of Porous Media*, Springer Verlag.
8 Kaviany, M. (1995) *Principles of Heat Transfer in Porous Media*, 2nd edn, Springer Verlag.
9 Oertel, H. (ed.) (2004) *Prandtl's Essentials of Fluid Mechanics*, Springer Verlag.
10 Vafai, K. (ed.) (2005) *Handbook of Porous Media*, CRC, Taylor & Francis Group.
11 Kolodziej, J.A. (1987) Review of application of boundary collocation methods in mechanics of continuous media. *Solid Mech. Arch.*, **12**, 187–231.
12 Kolodziej, J.A. and Strek, T. (2001) Analytical approximations of the shape factors for conductive heat flow in circular and regular polygonal cross-sections. *Int. J. Heat Mass Transfer*, **44**, 999–1012.
13 Kolodziej, J.A., Uscilowska, A. and Cialkowski, M. (2002) Semi-analytical approximations of the laminar friction coefficients for flow in conduits with polygonal cross-section. *Acta Mech.*, **158**, 127–144.
14 Iovane, G., Lifanov, I.K. and Sumbatyan, M.A. (2003) On direct numerical treatment of hypersingular integral equations arising in mechanics and acoustics. *Acta Mech.*, **162**, 99–110.
15 Lifanov, I.K. (1996) *Singular Integral Equations and Discrete Vortices*, VSP.
16 Mityushev, V. and Rogosin, S. (1999) *Constructive Methods for Linear and Nonlinear Boundary Value Problems for Analytic Functions. Theory and Applications, Monographs and Surveys in Pure and Applied Mathematics*, Chapman & Hall/CRC,**108**.
17 Bakhvalov, N. and Panasenko, G. (1989) *Mathematical Problems in the Mechanics of Composite Materials*, Kluwer Academic Publishers.
18 Bensoussan, A. Lions, J.-L. and Papanicolaou G. (1978) *Asymptotic Analysis of Periodic Structures*, North-Holland.
19 U. Hornung (ed.) (1997). *Homogenization and Porous Media (Interdisciplinary Applied Mathematics)*, Springer Verlag.
20 Jikov, V.V., Kozlov, S.M. and Olejnik, O.A. (1994) *Homogenization of Differential Operators and Integral Functionals*, Springer-Verlag.

21 Courant, R. and Hilbert, D. (1962) *Methods of Mathematical Physics*, Interscience Publishers.

22 Günter, N.M. (1967) *Potential Theory and its Applications to Basic Problems of Mathematical Physics*, Ungar.

23 Landkof, N.S. (1972) *Foundation of Modern Potential Theory*, Springer Verlag.

24 Borsuk, M. and Kondratiev, V. (2006) *Elliptic Boundary Value Problems of Second Order in Piecewise Smooth Domains*, Elsevier Science & Technology.

25 Mishuris, G.S. (1997) 2-D boundary value problems of thermoelasticity in a multi-wedge multi-layered region. I: Sweep method. II: System of integral equations. *Arch. Mech.*, **49**, 1103–1165.

26 Movchan, A.B. and Movchan, N.V. (1995) *Mathematical modelling of Solids with Nonregular Boundaries*, CRC.

27 Berenstein, C.A. and Gay, R. (1991) *Complex Variables. An Introduction*, Springer Verlag.

28 Mazya, V. (1985) *Sobolev Spaces*, Springer Verlag.

29 Miloh, T. and Benveniste, Y. (1999) On the effective conductivity of composites with ellipsoidal inhomogeneities and highly conducting interfaces. *Proc. R. Soc. Lond. A*, **455**, 2687–2706.

30 Mikhlin, S.G. (1964) *Integral Equations and Their Applications to Certain Problems in Mechanics, Mathematical Physics and Technology*, 2nd rev. edn, Macmillan.

31 Gakhov, F.D. (1977) *Boundary Value Problems*, 3rd edn, Nauka, ([in Russian] Engl transl of 1st edn: Pergamon Press 1966).

32 Bergman, D.J. (1978) The dielectric constants of a composite material – a problem in classical physics. *Phys. Rep. C*, **43**, 377–407.

33 Bergman, D.J. (1985) Bulk physical properties of composite media, in: *Les methodes de l'homogeneisation: theorie et applications en physique*, pp. 1–128. Volume **57** of Collection de la Direction des etudes et recherches d' Electricite de France, Session qui s'est tenue au Centre du Breau-sans-Nappe, du 27 juin au 13 juillet Eyrolles, 1983, Paris.

34 Bojarski, B. (1960) On generalized Hilbert boundary value problem. *Soobsch. AN GruzSSR*, **25**, 385–390 [in Russian].

35 Mikhailov, L.G. (1970) *New Class of Singular Integral Equations and its Applications to Differential Equations with Singular Coefficients*, AN TadzhSSR, 1963 [in Russian]; Engl. transl.:Akademie-Verlag.

36 Mityushev, V. (2005) R-linear problem on torus and its application to composites. *Complex Variables*, **50**, 621–630.

37 Adler, P.M. and Mityushev, V.M. (2001) Boundary value problems in porous media mechanics. *Proc. Inst. Math. Minsk*, **9**, 7–15.

38 Borisenko, A.I. and Tarapov, I.E. (1979) *Vector and Tensor Analysis with Applications*, Dover, New York.

39 Albin, N., Cherkaev, A. and Nesi, V. Multiphase laminates of extremal effective conductivity in two dimensions J. Mech. Phys. Solids (to appear).

40 Landauer, R. (1978) Electrical conductivity in inhomogeneous media, in: *Electrical Transport and Optical Properties of Inhomogeneous Media. AIP Conf. Proc.*, 2–43.

41 Markov, K.Z. (1999) *Heterogeneous Media: Modelling and Simulation*, Available online at http://www.fmi.uni-sofia.bg/fmi/contmech/kmarkov/pub/survey.pdf. K. Z. Markov and L. Preziosi, (eds.), Birkhäuser, pp. 1–162.

42 Zimmerman, R.W. (1996) Effective Conductivity of a two-dimensional medium containing elliptical inclusions. *Proc. R. Soc. Lond. A*, **452**, 1713–1727.

43 Movchan, A.B., Movchan, N.V. and Poulton, C.G. (2002) *Asymptotic Models of Fields in Dilute and Densely Packed Composites*, Imperial College Press.

44 Bruggeman, D.A.G. (1935) Berechnung verschiedener physikalischer Konstanten von heterogenen Substanzen. I. Dielektrizitätskonstanten und Leitfähigkeiten der Mischkörper aus isotropen Substanzen. *Ann. Phys. Lpz.*, **24**, 636–679.

45 Kirkpatric, S. (1973) Percolation and conduction. *Rev. Mod. Phys.*, **45**, 574–588.

46 Mori, T. and Tanaka, K. (1973) Average stress in the matrix and average elastic energy of materials with misfitting inclusions. *Acta Metal.*, **21**, 571–574.

47 Adler, P.M. and Mityushev, V. (2003) Effective medium approximations and exact formulas for electrokinetic phenomena in porous media. *J. Phys. A. Math. Gen.*, **36**, 391–404.

48 Keller, J.B. (1964) A theorem on the conductivity of a composite medium. *J. Math. Phys.*, **5**, 548–549.

49 Mathéron, G. (1967) *Eléments pour théorie des milieux poreux*, Masson.

50 Dykhne, A.M. (1971) Conductivity of a two-dimensional two-phase system. *Sov. Phys. JETP*, **32**, 63–65.

51 Adler, P., Malevich, A.E. and Mityushev, V. (2004) Macroscopic diffusion on a class of surfaces. *Phys. Rev. E*, **69**, 011607.

52 Shvidler, M.I. (1983) Effective conductivity of two-dimensional anisotropic media. *Soviet Physics – JETP*, **57**, 688–690.

53 Fel, L.G., Machavariani, V.Sh. and Bergman, D.J. (2000) Isotropic conductivity of two-dimensional three-component symmetric composites. *J. Phys. A: Math. Gen.*, **33**, 6669–6681.

54 Fel, L.G. (2002) Piezoelectricity and piezomagnetism: duality in two-dimensional checkerboards. *J. Math. Phys.*, **43**, 2606–2609.

55 Fel, L.G. On Keller theorem for anisotropic media, arXiv.org: cond-mat/0201319.

56 Indelman, P. (2002) On mathematical models of average flows in heterogeneous formations. *Transp. Porous Media*, **48**, 209–224.

57 Noetinger, B. (1994) The effective permeability of a heterogeneous porous medium. *Transp. Porous Media*, **15**, 99–127.

58 Mityushev, V. and Zhorovina, T.N. (1996) Exact solution to the problem on electrical field in doubly periodic heterogeneous media.Proc. Int. Conf. Boundary Value Problems, Special Functions and Fractional Calculus, Minsk, Belarus, 16–20 February pp.237–243.

59 Obnosov Yu, V. (1996) Exact solution of a boundary-value problem for a regular checkerboard field. *Proc. R. Soc. Lond. A*, **452**, 2423–2442.

60 Craster, R.V. and Obnosov Yu, V. (2001) Four phase periodic composites. *SIAM J. Appl. Math.*, **61**, 1839–1856.

61 Craster, R.V. and Obnosov Yu, V. (2001) Checkerboard composites with separated phases. *J. Math. Phys.*, **42**, 5379–5388.

62 Craster, R.V. and Obnosov Yu, V. (2004) A three-phase tessellation: solution and effective properties. *Proc. R. Soc. London A*, **460**, 1017–1037.

63 Craster, R.V. and Obnosov Yu, V. (2006) A model four-phase square checkerboard structure. *Quart. J. Mech. Appl. Math.*, **59**, 1–27.

64 Obnosov Yu, V. (1996) Solution of the problem of R-linear conjunction of the aggregates theory for a three-component medium. *Russian Mathematics*, **40**, 63–72.

65 Obnosov Yu, V. (1999) Periodic heterogeneous structures: new explicit solutions and effective characteristics of refraction of an imposed field. *SIAM J. Appl. Math.*, **59**, 1267–1287.

66 Kozlov, S.M. and Vucans, J. (1992) Explicit formula for effective thermoconductivity on the quadratic lattice structure. *C.R. Acad. Sci. Paris, Ser. I*, **314**, 281–286.

67 Borcea, L. and Papanicolaou, G. (1998) Network approximation for transport properties of high contrast materials. *SIAM J. Appl. Math.*, **58** (2), 501–539.

68 Berlyand, L.V., Gorb, Y. and Novikov, A. (2004) Discrete network approximation for highly-packed Composites with irregular geometry in three dimensions. in *Lecture Notes in Computational Science and Engineering* (eds B., Engquist P., Lotstedt and Runborg, O), Springer Verlag.

69 Berlyand, L.V. and Kolpakov, A.G. (2001) Network approximation in the limit of small interparticle distance of the effective properties of high contrast random dispersed composite. *Arch. Rational Mech.*, **159**, 179–227.

70 Berlyand, L.V. and Novikov, A. (2002) Error of the network approximation for densely packed composites with irregular geometry. *SIAM J. Mathmat. Anal.*, **34**, 385–408.

71 Gorb, Y. and Berlyand, L. (2005) Asymptotics the effective conductivity of composites with closely spaced inclusions of optimal shape. *Quart. J. Mech. Appl. Math.*, **58**, 83–106.

72 Keller, J.B. (1963) Conductivity of medium containing a dense array of perfectly conducting spheres or cylinders. *J. Appl. Phys.*, **34** (4), 991–993.

73 Kolpakov, A.G. (2006) Asymptotic imaging and network approximations of densly packed particles. *Inzhenerno-Fizicheskij Zhurnal*, **79**, 39–47 [in Russian].

74 Akhiezer N.I. (1990) *Elements of Theory of Elliptic Functions*, Nauka, 1970 (in Russian); Engl. transl. AMS.

75 Wang, Z.X. and Guo, D.R. (1989) *Special Functions*, World Scientific.

76 Weil, A. (1976) *Elliptic Functions According to Eisenstein and Kronecker*, Springer-Verlag.

77 Rayleigh, J.W. (1892) On the influence of obstacles arranged in rectangular order upon the properties of a medium. *Phil. Mag.*, **32**, 481–491.

78 McPhedran, R.C. and Milton, G. (1987) Transport properties of touching cylinder pairs and of the square array of touching cylinders. *Proc. R. Soc. Lond. A*, **411**, 313–326.

79 McPhedran, R.C., Poladian, L. and Milton, G. (1988) Asymptotic studies of closely spaced highly conducting cylinders. *Proc. R. Soc. Lond. A*, **415**, 185–196.

80 Mityushev V. (1997) Transport properties of regular arrays of cylinders. *ZAMM*, **77**, 115–120.

81 Rylko, N. (2000) Transport properties of the regular array of highly conducting cylinders. *J. Eng. Math.*, **38**, 1–12.

82 Mityushev, V. (1997) Transport properties of finite and infinite composite materials and Rayleigh's sum. *Arch. Mech.*, **49**, 345–358.

83 Mityushev V. (1999) Transport properties of two-dimensional composite materials with circular inclusions. *Proc. R. Soc. London A*, **455**, 2513–2528.

84 Berdichevskij, V.L. (1983) *Variational Principles of Continuum Mechanics*, Nauka, [in Russian].

85 Huang, J. (1999) Integral representations of harmonic lattice sums. *J. Math. Phys.*, **40**, 5240–5246.

86 Berlyand, L. and Mityushev, V. (2001) Generalized Clausius–Mossotti formula for random composite with circular fibers. *J. Statist. Phys.*, **102**, 115–145.

87 Berlyand, L. and Mityushev, V. (2005) Increase and decrease of the effective conductivity of a two phase composites due to polydispersity. *J. Statist. Phys.*, **118**, 481–509.

88 Mityushev, V.V. (1997) Functional equations and its applications in mechanics of composites. *Demonstr. Math.*, **30**, 64–70.

89 Mityushev V. (1998) Steady heat conduction of the material with an array of cylindrical holes in the non-linear case. *IMA J. Appl. Math.*, **61**, 91–102.

90 Mityushev, V. (2001) Transport properties of doubly periodic arrays of circular cylinders and optimal design problems. *Appl. Math. Optimization*, **44**, 17–31.

91 Mityushev, V. (2006) Representative cell in mechanics of composites and generalized Eisenstein–Rayleigh sums. *Complex Variables*, **51**, 1033–1045.

92 Artola, M. and Duvaut, G. (1982) Un resultat d'homogeneisation pour une classe de problemes de diffusion non lineaires stationnaires. *Annales Faculté des Sciences Toulouse*, **IV**, 1–27.

93 Galka, A., Telega, J.J. and Tokarzewski, S. (1997) Nonlinear transport equation and macroscopic properties of microheterogeneous media. *Arch. Mech.*, **49**, 293–319.

94 Bruno, O. (1991) The effective conductivity of strongly heterogeneous composites. *Proc. R. Soc. London A*, **433**, 353–381.

95 Helsing, J. (1994) Bounds on the shear modulus of suspensions by interface method. *J. Mech. Phys. Solid*, **42**, 1123–1138.

96 Mityushev, V. and Adler, P.M. (2002) Longitudial permeability of a doubly periodic rectangular array of circular cylinders. *I ZAMM*, **82**, 335–345.

97 Movchan, A.B., Nicorovici, N.A. and McPhedran, R.C. (1997) Green's tensors and lattice sums for elastostatics and elastodynamics. *Proc. R. Soc. London A*, **453**, 643–662.

98 Perrins, W.T., McKenzie, D.R. and McPhedran, R.C. (1979) Transport properties of regular array of cylinders. *Proc. R. Soc. Lond. A*, **369**, 207–225.

99 Ponte Castañeda, P. Suquet, P.(1998) Nonlinear composites. in *Advances in Applied Mechanics* (eds E. van der Giessen *et al.*,), Academic Press, Vol. 34 pp. 171–302.

100 Ponte Castañeda, P. and Zaidman, M. (1996) The finite deformation of nonlinear composite materials. I. Instantaneous constitutive relations. II: Evolution of the microstructure. *Int. J. Solids Struct.*, **33**, 1271–1303.

101 Schulgasser, K. (1977) Bounds on the conductivity of statistically isotropic polycrystal. *J. Phys. C*, **10**, 407–417.

102 Szczepkowski, J., Malevich, A.E. and Mityushev, V. (2003) Macroscopic properties of similar arrays of cylinders. *Quart. J. Appl. Math. Mech.*, **56**, 617–628.

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