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REAL ANALYTICITY OF AN OPERATOR ASSOCIATED TO THE SCHWARZ OPERATOR FOR MULTIPLE CONNECTED DOMAINS

Abstract

The paper deals with the regularity of the operator which assigns to the boundary curve $\phi$ of a multiply connected domain and to the boundary data $f$ the boundary values of a single-valued restriction of the (possibly multi-valued) solution $F$ of the Schwarz problem. The real analyticity of such an operator with respect to $\phi$ and $f \circ \phi$ is obtained by applying the Implicit Function Theorem to the operator associated to the boundary value problem for $F'$.

1. Introduction

In this paper we consider the operator which maps each pair $(\phi, p)$ of $n$ simple counter clock-wise oriented disjoint smooth curves $\phi \equiv (\phi_1, \ldots, \phi_n)$ with $\phi_j : \mathbb{T} \to \mathbb{C}$, and $\text{cl}(\mathbb{T} \setminus \phi_j) \cap \text{cl}(\phi_k)) = \emptyset, j \neq k$, (here $\mathbb{T}$ is the unit circle, and $I[\phi_j], E[\phi_j]$ denote respectively the bounded and the unbounded components of $\mathbb{C} \setminus \phi_j(\mathbb{T})$) and of $n$ continuous real valued functions $p \equiv (p_1, \ldots, p_n)$ with $p_j : \mathbb{T} \to \mathbb{R}$, to the boundary values $F|_\phi \equiv (F|_{\phi_1}, \ldots, F|_{\phi_n})$ of the solution of the following modification of the classical Schwarz problem (see e.g. [4, p. 208]) in a multiply connected exterior domain $D = \bigcap_{j=1}^{n} E[\phi_j] \equiv E[\phi]$

\[
\begin{cases}
F \in \mathcal{H}(E[\phi]) \cap \mathcal{C}^0(\text{cl} E[\phi]),
\Re F|_{\phi_j(\mathbb{T})} = p_j \circ \phi_j^{(-1)} + \lambda_j, \quad \text{for some } \lambda_j \in \mathbb{R} \quad j = 1, \ldots, n.
\end{cases}
\]

(1.1)

Our aim is to study the regularity of an operator associated to the derivative of a normalized solution $F$ of (1.1). The study of the regularity of certain nonlinear operators with respect to functional variables is quite intensive in the recent years (see e.g. [9], [10] and references therein). These results can be applied to the perturbation analysis of the boundary value problems involving the studied operators (see e.g. [8]). The real analyticity in Schauder spaces of the operator which maps $(\phi, p)$ to $F \circ \phi$, where $F$ is the unique normalized single-valued solution of (1.1), has been proved in [18] for $n = 1$ and in [19] for general $n$.

Since the Schwarz problem and its modifications can be applied to the study of many two-dimensional physical models, it has been considered by several authors (see e.g. [12], [15], [16], [21], [23]). Among these applications we have to mention the elasticity problems (see e.g. [12, Ch. IX]), problems of the fluid mechanics (see e.g. [22], and those with free boundary as in [16], [20]), and the composite material problems (see e.g. [14], [15]).

The solution of the classical Schwarz problem for a multiply connected domain (i.e. (1.1) with $\lambda_j = 0, j = 1, \ldots, n$) is in general a multi-valued holomorphic function. A modification of this problem had been proposed (see e.g. [12, p. 150]). Besides, the classical Schwarz problem as well as its modification possess an explicit solution only in special cases (these are the celebrated Villat’s formula for concentric annulus (see e.g. [1, p. 173]), and the formula for the solution in a multiply connected circular domain (see e.g. [15, p. 158-159])).

In our paper we look for an analyticity result which implies the analyticity of $(\phi, p) \mapsto F \circ \phi$ whenever $F$ (or a suitable restriction of $F$) is single-valued, taking into account the above difficulties.
A natural way to do that is to reduce (1.1) to the problem for its single-valued derivative \( W(z) = \frac{\partial \mathcal{E}}{\partial z} \), i.e.

\[
\begin{align*}
\text{W} & \in \mathcal{H}(\mathbb{E}[\phi]) \cap C^0(\partial \mathbb{E}[\phi]), \\
\lim_{z \to \infty} W(z) & = 0, \\
\text{Re} \left\{ \int t\phi_j(t)W(\phi_j(t)) \right\} & = itp_j(t), \ t \in \mathbb{T}, \ j = 1, \ldots, n, \\
\text{Im} \left\{ \int \phi_j W(\xi) d\xi \right\} & = c_j, \ j = 1, \ldots, n,
\end{align*}
\]

(1.2)

where \( c \equiv (c_1, \ldots, c_n) \in \mathbb{R}^n \) is an \( n \)-vector which makes (1.2) uniquely solvable. We show that (1.2) is a well-posed problem for all choice of \((\phi, p, c)\) and we prove the real analyticity in the Schauder spaces of

\[
W_* : (\phi, p, c) \to W_{\phi} \circ \phi
\]

by applying the Implicit Function Theorem to a suitable operator associated to (1.2). The regularity results of [18] and [19] follow from the regularity of \( W_* \) by taking \( c = 0 \) and by a simple integration. Moreover the analyticity of \( W_* \) implies immediately that the velocity field of a stationary potential fluid past \( n \) obstacles \( \partial \mathbb{E}[\phi_j] \) \( (j = 1, \ldots, n) \) approaching a given flow \( V_\infty \) at infinity depends real analytically on \( V_\infty \), on the shapes \( \phi_j \) of the obstacles and on the circulations around them.

Our paper is organized as follows. Necessary notations and auxiliary results are presented in Section 2. Section 3 is devoted to the detailed study of the problems (1.1), (1.2) and their connections to the Schwarz boundary value problem. The proof of the real analyticity of the boundary operator corresponding to the solution of problem (1.2) is given in Section 4.

2. Notation and Auxiliary Results

For standard definitions of Complex Analysis we refer to Ahlfors [2] (e.g. we denote by \( \mathbb{U} \) the open unit disc in \( \mathbb{C} \), by \( \mathbb{T} \) the boundary of \( \mathbb{U} \), and by \( cl \mathbb{U} \) the closure of \( \mathbb{U} \). For all \( a \in \mathbb{C} \), \( r \in (0, +\infty] \), \( a + r\mathbb{T} \) denotes the circle \( \{ z \in \mathbb{C} : |z - a| = r \} \) (counter clockwise oriented if it is the domain of integration), and \( a + r\mathbb{U} \) denotes the disc \( \{ z \in \mathbb{C} : |z - a| < r \} \).

Note that a regular curve is often defined as an equivalence class of regular parametrizations. However, we need to distinguish the different parametrizations. Thus we define a curve of class \( C^1 \) to be a map \( \phi \) of class \( C^1 \) from the boundary \( \mathbb{T} \) of the unit disk \( \mathbb{U} \) to \( \mathbb{C} \). By a simple curve of class \( C^1 \), we understand an injective map of class \( C^1 \) from \( \mathbb{T} \) to \( \mathbb{C} \). Also, a curve \( \phi \) should not be confused with \( \phi(\mathbb{T}) \).

We use here special notations \( \mathcal{C}_m(K, \mathbb{C}) \), \( \mathcal{C}^{m, \alpha}(K, \mathbb{C}) \) for the spaces of \( m \)-times differentiable functions of \( K \) (with \( ||f||_{\mathcal{C}_m(K, \mathbb{C})} \equiv \sum_{s=0}^{m} \sup |f^{(s)}| \) and of those having \( \alpha \)-Hölder continuous derivative of \( m \)-th order in \( K \) (with \( ||f||_{\mathcal{C}^{m, \alpha}(K, \mathbb{C})} \equiv ||f||_{\mathcal{C}_m(K, \mathbb{C})} + \sum \sup |f^{(m)}(t_1) - f^{(m)}(t_2)|/|t_1 - t_2|^\alpha : t_1, t_2 \in K, t_1 \neq t_2 \) ), \( K \) being a compact subset of \( \mathbb{C} \) (e.g. \( K = \phi(\mathbb{T}) \)). Thus the sign \( * \) means that all derivatives are taken along \( K \). For standard definitions of Calculus in Normed Spaces we refer e.g. to Berger [3].

By a simple contradiction argument, it can be readily verified that the following holds (cf. [8, p. 1201], [7, p. 124]).
Lemma 2.1.  The set
\[ Z \equiv \left\{ \phi \in C^1_\alpha(T, \mathbb{C}) : \inf \left\{ \frac{|\phi(s) - \phi(t)|}{|s-t|} : s, t \in T, s \neq t \right\} > 0 \right\} \]
coincides with the set of simple curves $\phi$ of class $C^1_\alpha(T, \mathbb{C})$ with nowhere vanishing $\phi'$.  The sets $Z$ and its subset $Z^+ (Z^-)$ of all positively (negatively) oriented curves are open in $C^1_\alpha(T, \mathbb{C})$. Moreover the set
\[ Z^+ \equiv \left\{ \phi \equiv (\phi_1, \ldots, \phi_n) \in (Z^+)^n : \phi_k(T) \subseteq \bigcap_{j \neq k} E[\phi_j], k = 1, \ldots, n \right\} \]
is open in $(C^1_\alpha(T, \mathbb{C}))^n$.

If $f = (f_1, \ldots, f_n)$ and $\phi = (\phi_1, \ldots, \phi_n)$ and all $f_j \circ \phi_j$ are meaningful $f \circ \phi$ denotes the $n$-vector $(f_j \circ \phi_j)_{j=1}^n$. Correspondingly $f \phi \equiv (f_j \phi_j)_{j=1}^n$, $\text{Re} f \equiv (\text{Re} f_j)_{j=1}^n$, $\int f \equiv (\int f_j)_{j=1}^n$ and, if all $\phi_j$’s are invertible, $\phi^{-1} \equiv (\phi_j^{-1})_{j=1}^n$.

We are now ready to state the following, which collects a few facts which we need on the spaces $C^{m,\alpha}_s(K, \mathbb{C})$. For a proof and for appropriate references, we refer to [9, Lems. 2.7, 2.8].

Lemma 2.2. Let $m \in \mathbb{N}$, $\alpha, \beta \in [0, 1]$, $\phi \in Z$, $L = \phi(T)$. Then the following statements hold.

(i) $C^{m+1}_s(L, \mathbb{C})$ is continuously imbedded in $C^{m,\alpha}_s(L, \mathbb{C})$. If $\alpha < \beta$, then $C^{m,\beta}_s(L, \mathbb{C})$ is continuously imbedded in $C^{m,\alpha}_s(L, \mathbb{C})$.

(ii) The pointwise product is continuous in the Banach space $C^{m,\alpha}_s(L, \mathbb{C})$.

(iii) The reciprocal map $R$ in $C^{m,\alpha}_s(L, \mathbb{C})$, which maps a nonvanishing function $f$ to its reciprocal, is complex analytic from the open subset $C^{m,\alpha}_s(L, \mathbb{C} \setminus \{0\})$ of $C^{m,\alpha}_s(L, \mathbb{C})$ to itself.

(iv) Let $\phi_1 \in Z$, $L_1 = \phi_1(T)$. If $f \in C^{m,\alpha}_s(L_1, \mathbb{C})$ and if $g \in C^{m,\beta}_s(L, L_1)$, then $f \circ g \in C^{m,\gamma}_{s,\alpha,\beta}(L, \mathbb{C})$ with $\gamma = \gamma_{\alpha, \beta} = \alpha \beta$ and $\gamma_{m, \alpha, \beta} = \min\{\gamma_{0, \alpha, \beta}, \gamma_{0, \alpha, \beta}, \gamma_{m, \alpha, \beta}\}$ if $m > 0$. Moreover the operator $h \mapsto h \circ g$ is linear and continuous from $C^{m,\alpha}_s(L_1, \mathbb{C})$ to $C^{m,\gamma}_{s,\alpha,\beta}(L, \mathbb{C})$.

(v) Let $m \geq 1$. If $g \in C^{m,\alpha}_s(L, \mathbb{C})$ is injective and satisfies condition $g'(\xi) \neq 0$, for all $\xi \in L$, then $g^{-1} \in C^{m,\alpha}_s(g(L), L)$.

The following Theorem collects known facts related to singular integrals with Cauchy kernels and to Cauchy type integrals.

Theorem 2.3. Let $\alpha \in [0, 1]$, $m \in \mathbb{N}$, $\phi \in C^{1,\alpha}_s(T, \mathbb{C}) \cap Z^+$, $L = \phi(T)$. Let $I_L$ be the identity operator in $C^{1,\alpha}_s(L, \mathbb{C})$. Then the following statements hold.

(i) For all $f \in C^{m,\alpha}_s(L, \mathbb{C})$, the singular integral
\[ \mathbf{S}_f(\phi) = \text{P.V.} \frac{1}{2\pi i} \int_\phi \frac{f(\xi)}{\xi - \tau} d\xi = \text{P.V.} \frac{1}{2\pi i} \int_\tau \frac{f(\phi(\xi))\phi'(\xi)}{\phi(\xi) - \tau} d\xi, \quad \forall \tau \in L, \]
exists in the sense of the principal value, and $\mathbf{S}_f(\phi) \in C^{m,\alpha}_s(L, \mathbb{C})$. The operator $\mathbf{S}_\phi$ is linear and continuous from $C^{m,\alpha}_s(L, \mathbb{C})$ to itself and satisfies $\mathbf{S}_\phi \circ \mathbf{S}_\phi = I_L$.

Moreover $(\mathbf{S}_f)^* = \mathbf{S}_{f^*}$ for all $f \in C^{1,\alpha}_s(L, \mathbb{C})$.

(ii) For all $f \in C^{m,\alpha}_s(L, \mathbb{C})$, the function $C_f(\phi)$ of $L$ to $\mathbb{C}$ defined by
\[ C_f(\phi)(z) = \frac{1}{2\pi i} \int_\phi \frac{f(\xi)}{\xi - z} d\xi, \quad \forall z \in \mathbb{C} \setminus L, \]
The operator $\lim_{\tau \to -\infty} C_{\phi}^+ [f](\tau)$ admits a continuous extension to $\overline{\Omega}[\phi]$, which we denote by $C_{\phi}^+ [f]$, and the function $C_{\phi}^- [f] |_{\overline{\Omega}[\phi]}$ admits a continuous extension to $\overline{\Omega}[\phi]$, which we denote by $C_{\phi}^- [f]$. Then we have $C_{\phi}^+ [f] \in H(\overline{\Omega}[\phi]) \cap \mathcal{C}^{m,\alpha}(\overline{\Omega}[\phi], \mathbb{C})$, $C_{\phi}^- [f] \in H(\overline{\Omega}[\phi]) \cap \mathcal{C}^{0}(\overline{\Omega}[\phi], \mathbb{C}) \cap \mathcal{C}^{m,\alpha}_*(L, \mathbb{C})$, and the Sokhotsky-Plemelj formulas $C_{\phi}^+ [f](\tau) = \pm \frac{i}{2} f(\tau) + \frac{1}{2} S_\phi [f](\tau)$ for all $\tau \in L$ hold. Furthermore, $\lim_{z \to \infty} C_{\phi}^- [f](z) = 0$.

(iii) The function $f \in \mathcal{C}^{m,\alpha}_*(L, \mathbb{C})$ satisfies equation $(I_L - S_\phi)[f] = 0$, if and only if there exists a function $F \in H(\overline{\Omega}[\phi]) \cap \mathcal{C}^{m,\alpha}(\overline{\Omega}[\phi], \mathbb{C})$ such that $F(\tau) = f(\tau)$, for all $\tau \in L$. The function $F$, if it exists, is unique.

(iv) Let $\phi \in \mathbb{Z}^+$. A vector-function $\mathbf{f} \equiv (f_1, \ldots, f_n) \in \prod_{j=1}^n \mathcal{C}^{m,\alpha}_{\phi}(\phi_j(\mathbb{T}), \mathbb{C})$ satisfies the system

\[(2.2) \quad (I_{\phi_j(\mathbb{T})} + S_{\phi_j}) [f_j](t_j) + 2 \sum_{k=1, k \neq j}^n C_{\phi_k}^- [f_k](t_j) = 0, \quad t_j \in \phi_j(\mathbb{T}), \quad j = \ldots, n,
\]

if and only if there exists $F \in H(\mathbb{E}[\phi_0]) \cap \mathcal{C}^0(\overline{\Omega}[\phi], \mathbb{C}) \cap \mathcal{C}^{m,\alpha}_*(\overline{\Omega}[\phi_j(\mathbb{T}), \mathbb{C})$ null at infinity such that $F(\phi_j(\mathbb{T})) = f_j$, $j = 1, \ldots, n$. If such $F$ exists then it is unique and $F(z) = -\sum_{j=1}^n C_{\phi_j}^+ [f_j](z)$ for all $z \in \mathbb{E}[\phi]$.

The following Lemma yields a functional equation which characterizes the boundary values of a holomorphic function in an exterior of multiply connected domain.

**Lemma 2.4.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in [0, 1[. Let $\phi \equiv (\phi_1, \ldots, \phi_n) \in \mathcal{C}^{m,\alpha}_*(\mathbb{T}, \mathbb{C})^n \cap \mathbb{Z}^+$ and $\mathbf{q} \equiv (q_1, \ldots, q_n) \in \prod_{j=1}^n \mathcal{C}^{m,\alpha}_*(\phi_j(\mathbb{T}), \mathbb{C})$. Let $S_{\phi_j}[\cdot]$ be the singular integral operator on $\phi_j$ introduced by formula (2.1). Let $P(\phi, \cdot)$ be the operator of $\prod_{j=1}^n \mathcal{C}^{m,\alpha}_*(\phi_j(\mathbb{T}), \mathbb{C})$ to itself defined by

\[
P(\phi, \mathbf{q}) = (P_j(\phi, \mathbf{q}))_j^{n-1} = \left( \frac{q_j(t_j)}{2} + \frac{S_{\phi_j}[q_j](t_j)}{2} + \sum_{k=1, k \neq j}^n \frac{1}{2\pi i} \int_{\phi_k} \frac{q_k(\xi) d\xi}{\xi - t_j} \right)_{j=1}^n.
\]

Then the following statements hold.

(i) The operator $P(\phi, \cdot)$ is a continuous projection. In particular

\[\text{Im} \left( P(\phi, \cdot) \right) = \text{Ker} \left( I - P(\phi, \cdot) \right).\]

(ii) The equality $P(\phi, \mathbf{q}) = \mathbf{0}$ holds if and only if $\mathbf{q}$ are boundary values of a (unique) holomorphic function $Q$ of $\mathbb{E}[\phi]$ null at infinity. Moreover for all $\mathbf{q} \in \prod_{j=1}^n \mathcal{C}^{m,\alpha}_*(\phi_j(\mathbb{T}), \mathbb{C})$ the function $P_j(\phi, \mathbf{q})$ is the boundary value of a holomorphic function of $1[\phi_j]$ for all $j = 1, \ldots, n$.

(iii) Let $\phi, \phi_0 \in \mathcal{C}^{m,\alpha}_*(\mathbb{T}, \mathbb{C})^n \cap \mathbb{Z}^+$. Then the equality

\[
(2.3) \quad P \left[ \phi_0, P(\phi, \mathbf{q}) \circ (\phi \circ (\phi_0)^{-1}) \right] = 0
\]

holds if and only if $P(\phi, \mathbf{q}) = \mathbf{0}$.

**Proof.** Statements (i) and (ii) follows from the Cauchy formula and from the properties of the Cauchy type integral described in Theorem 2.3. We prove (iii) by using a basic property of the boundary value problems with shift. By (ii) $P_j(\phi, \mathbf{q})$ extends to a holomorphic map in $1[\phi_j]$ for all $j = 1, \ldots, n$. By (2.3) $P(\phi, \mathbf{q})$ composed with the generalized shift $\phi \circ (\phi_0)^{-1}$ extends to a holomorphic map $\mathbb{E}[\phi_0]$ null at infinity. Assume that $P(\phi, \mathbf{q}) \neq \mathbf{0}$. By arguing as in [4, pp. 123-124] concerning the
case $n = 1$ the product powers $\{P[\phi, q]^s\}_{s \in \mathbb{N} \setminus \{0\}}$ would be an independent family belonging to the kernel of a Fredholm operator of 0 index. The contradiction yields $P[\phi, q] = 0$.

The following Lemma presents the conditions of single-valuedness of functions holomorphic in a multiply connected domain (see e.g. [2], [5])

**Lemma 2.5.** Let $D$ be a multiply connected domain exterior to $n$ simple smooth nonintersecting curves $L_j$, $j = 1, \ldots, n$. Let $F$ be a function holomorphic (but possibly multi-valued) in the domain $D$ with single-valued derivative $F'$ continuous up to the boundary of $D$.

The function $F$ is single-valued in $D$ if and only if the circulation of $F'$ and $iF'$ along all boundary curves vanishes, i.e.

$$\int_{L_j} F'(\xi)d\xi = 0, \quad j = 1, \ldots, n.$$

We now observe the following result about the Riemann map of multiply connected exterior domain.

**Lemma 2.6.** Let $m, n \in \mathbb{N} \setminus \{0\}$, $\alpha \in [0, 1]$, let $Z^+$ be as in Lemma 2.1 and let $\phi \in \mathcal{C}^{m, \alpha}_{\sigma} (\mathbb{T}, \mathbb{C})^n \cap Z^+$. Then there exist $a \in \mathbb{C}^n$, $r \in \mathbb{R}^n_+$ with $|a_k - a_j| > r_k + r_j$ for all $k, j \in \{1, \ldots, n\}, k \neq j$, and a conformal map $g$ of $D_{a, r} \equiv \bigcap_{j=1}^n E[a_j + r_j \mathbb{T}]$ onto $E[\phi] \equiv \bigcap_{j=1}^n E[\phi_j]$ with $\lim_{z \to \infty} g'(z) \in \mathbb{C} \setminus \{0\}$. Moreover $g$ can be uniquely extended to a homeomorphism of $\text{cl} D_{a, r}$ to $\text{cl} E[\phi]$ and the restriction of $g$ on the boundary belongs to the corresponding Schauder space, i.e. $g|_{[a_j + r_j \mathbb{T}]} \in \mathcal{C}^{m, \alpha}_{\sigma} (a_j + r_j \mathbb{T}, \mathbb{C})$ for all $j \in \{1, \ldots, n\}$.

**Proof.** The existence is proved e.g. in [5, p. 235] (see also [6]). The regularity of the conformal map in the case $n = 2$ is shown in [11, Thm. 3.1] by using a Pommerenke suggestion and the Warschawski regularity result concerning $n = 1$ (cf. e.g., [17, Thms. 3.5, 3.6]). The proof of regularity of the conformal map of an arbitrary multiply connected domain can be completed by induction applying the same idea as in the case $n = 2$.  

3. The Schwarz Problem for Multiply Connected Domains and its Reduction

In this Section we consider a suitably normalized solution of the Schwarz problem and we prove the well-posedness of a boundary value problem for its derivative. By the latter problem we will introduce an operator whose regularity will be studied in the next Section.

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in [0, 1]$. Let $\phi = (\phi_1, \ldots, \phi_n) \in \mathcal{C}^{m, \alpha}_{\sigma} (\mathbb{T}, \mathbb{C})^n \cap Z^+$, $f = (f_1, \ldots, f_n) \in \prod_{j=1}^n \mathcal{C}^{m, \alpha}_{\sigma} (\phi_j(\mathbb{T}), \mathbb{R})$, $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$. We first consider the following variant of the Schwarz problem. It consists in finding a possibly multi-valued function $F$ holomorphic in $E[\phi]$, and continuous up to the boundary such that

$$\begin{align*}
\lim_{z \to \infty} F'(z) &= 0, \\
(Re F)|_{\phi_j(\mathbb{T})} &= f_j + \lambda_j, \quad \text{for some } \lambda_j \in \mathbb{R}, j = 1, \ldots, n, \quad (a) \\
\text{Im} \left\{ \int_{\phi_j} F'(\xi)d\xi \right\} &= c_j, \quad j = 1, \ldots, n. \quad (b)
\end{align*}
$$

(3.1)
We first observe that, by subtracting a suitable multi-valued solution, problem (3.1) is equivalent to the following single-valued Schwarz problem

\[
\begin{aligned}
    \tilde{F} &\in \mathcal{H}(\mathbb{E} [\phi]) \cap C^0(\text{cl} \mathbb{E} [\phi]), \\
    \lim_{\gamma \to \infty} \tilde{F}(z) &\in \mathbb{C}, \\
    \left( \text{Re} \tilde{F} \right)_{|_{\phi_j(\gamma)}} &= f_j + \lambda_j, \\
    \text{for some } \lambda_j &\in \mathbb{R}, \; j = 1, \ldots, n.
\end{aligned}
\]  

(3.2)

Indeed, let \( a = (a_1, \ldots, a_n) \in \mathbb{C}^n \), \( r = (r_1, \ldots, r_n) \in \mathbb{R}_+^n \) and \( D_{a, r} = \bigcap_{j=1}^n \mathbb{E}[a_j + r_j \mathbb{T}] \). Let \( g \) be the conformal mapping of \( D_{a, r} \) to \( \mathbb{E} [\phi] \) as in Lemma 2.6. We consider the multi-valued map of \( \text{cl} \mathbb{E} [\phi] \)

\[
F_0(z) = \sum_{k=1}^n \frac{c_k}{2\pi} \log \left( g^{-1}(z) - a_k \right).
\]

(3.3)

Whenever \( c \in \mathbb{R}^n \) the real part of \( F_0 \) is single-valued and satisfies the relation

\[
\text{Re} F_0(t_j) = \frac{c_j}{2\pi} \log r_j + \sum_{k=1, k \neq j}^n \frac{c_k}{2\pi} \log \left| g^{-1}(t_j) - a_k \right|
\]

for all \( t_j \in \phi_j(\mathbb{T}) \) and \( j = 1, \ldots, n \). By Lemmas 2.2 (iv), (v) and 2.6 (\( \text{Re} F_0 \mid_{\phi_j(\gamma)} \in C^{m, \alpha}_* (\phi_j, \mathbb{T}) \), \( \mathbb{R} \)). Moreover the function \( F_0 \) defined in (3.3), even for \( c \in \mathbb{R}^n \), has single-valued derivative and satisfies

\[
\int_{\phi_j} F'_0(\xi) d\xi = \sum_{k=1}^n \frac{c_k}{2\pi} \int_{\phi_j} \frac{g^{-1}'(\xi)}{(g^{-1}(\xi) - a_k)} d\xi = \sum_{k=1}^n \frac{c_k}{2\pi} \int_{\phi_j} \frac{1}{\zeta - a_k} d\zeta = ic_j,
\]

(4.1)

for all \( j = 1, \ldots, n \). Let \( F \) be a solution to (3.1). It follows that \( \text{Im} \{ \int_{\phi_j} (F - F_0)'(\xi) d\xi \} = 0 \) for all \( j = 1, \ldots, n \). To prove that \( F - F_0 \) is single-valued and solves (3.2), by Lemma 2.5 it remains to show that \( \text{Re} \{ \int_{\phi_j} F'(\xi) d\xi \} = 0 \) for all \( j = 1, \ldots, n \). To show it we observe the validity of the following

\[
\begin{aligned}
\text{Re} \left\{ \int_{\phi_j} F'(\xi) d\xi \right\} &= \int_0^{2\pi} \text{Re} \left\{ F'(\phi_j(e^{i\theta})) \phi_j'(e^{i\theta}) ie^{i\theta} \right\} d\theta \\
&= \int_0^{2\pi} \text{Re} \left\{ \frac{\partial}{\partial \theta} \{ F(\phi_j(e^{i\theta})) \} \right\} d\theta = \int_0^{2\pi} \left\{ \frac{\partial}{\partial \theta} \{ f_j(\phi_j(e^{i\theta})) \} \right\} d\theta = 0.
\end{aligned}
\]

In the next Proposition we summarize the considerations we have just done.

**Proposition 3.1.** Let \( m, n \in \mathbb{N} \setminus \{0\}, \alpha \in [0, 1] \). Let \( \phi \in C^{m, \alpha}_* (\mathbb{T}, \mathbb{C})^n \cap \mathbb{Z}^+ \) and let \( f \in \prod_{j=1}^n C^{m, \alpha}_* (\phi_j, \mathbb{T}), \ c \in \mathbb{R}^n \). Let \( F_0 \) be the multi-valued function defined in (3.3). Then \( F \) is a (possibly multi-valued) solution of (3.1) if and only if \( F - F_0 \) is a single-valued solution bounded at \( \infty \) of (3.2) associated to \( f \equiv f_j - (\text{Re} F_0)_{|_{\phi_j(\gamma)}} \) for all \( j = 1, \ldots, n \). Moreover \( F \) is single-valued if and only if \( c = 0 \).

We now consider the problem (3.2) by using the result of [15]. First of all we observe that a standard argument involving harmonic measures implies the unique solvability of (3.2) up to an arbitrary complex constant.
Lemma 3.2. Let \( m, n \in \mathbb{N} \setminus \{0\}, \alpha \in ]0, 1[ \). Let \( \phi \in C_{\ast}^{m,\alpha}(\mathbb{T}, \mathbb{C})^n \cap \mathbb{Z}^+ \) and let \( \tilde{f} \equiv (\tilde{f}_1, \ldots, \tilde{f}_n) \in \prod_{j=1}^n C_{\ast}^{m,\alpha}(\phi_j(\mathbb{T}), \mathbb{R}) \). We assume that \( \tilde{F} \) is a solution of (3.2) associated to \( \lambda \in \mathbb{R}^n \).

Then \( \tilde{F}_1 \) is another solution of (3.2) associated to a possibly different constant vector \( \lambda \in \mathbb{R}^n \) if and only if \( \tilde{F}_1 = \tilde{F} + c \) for certain constant \( c \in \mathbb{C} \).

Proof. By linearity of (3.2) it is enough to prove the constancy of a solution \( \tilde{F} \) to (3.2) with \( \tilde{f} = 0 \) and \( \lambda \in \mathbb{R}^n \). Let \( \alpha_j(z), j = 1, \ldots, n \), be the harmonic measures associated to the multiply connected domain \( E[\phi] \cup \{\infty\} \) (see e.g. [2, p. 254] or [15, p. 149]). By the unique solvability of the Dirichlet problem for the Laplace equation we have

\[
Re \tilde{F}(z) = \sum_{j=1}^n \lambda_j \alpha_j(z), \quad \sum_{j=1}^n \alpha_j(z) = 1, \quad \forall z \in \text{cl} E[\phi],
\]

and, in particular, \( \sum_{j=1}^n \lambda_j \alpha_j(z) \) has a single-valued conjugate in \( E[\phi] \cup \{\infty\} \). Then \( \sum_{j=1}^n (\lambda_j - \lambda_n) \alpha_j(z) \) has a single-valued conjugate in \( E[\phi] \cup \{\infty\} \). By a standard argument (see e.g. [2, p. 254]) it follows that \( \lambda_j = \lambda_n \) for all \( j = 1, \ldots, n \) and \( Re \tilde{F} = \lambda_n \) in \( \text{cl} E[\phi] \). Thus \( \tilde{F} \) is constant.

The existence part for problem (3.2) together with a description of the solution has been developed in [15, p. 149-159]. We will recall this result. Let \( a \in \mathbb{C}^n, r \in \mathbb{R}^n \), \( |a_k - a_j| > r_k + r_j, k \neq j \), and \( D_{\mathbf{a}, \mathbf{r}} \) be a multiply connected circular domain. Thus \( \partial D_{\mathbf{a}, \mathbf{r}} = \bigcup_{k=1}^n (a_k + r_k \mathbb{T}) \). Introduce the following mappings:

\[
(z^{k_1, \ldots, k_p}_{k_{p-1}, \ldots, k_1})(k_j) := \left(z^{(k_{p-1}, \ldots, k_1)}{k_j}\right)^*,
\]

where \( z^{k_j}(k_j) = \frac{z^{k_j}}{a_k} + a_k \) is the symmetry with respect to the k-th circle \( a_k + r_k \mathbb{T} \). Hence, \( z^{k_1, \ldots, k_p}_{k_{p-1}, \ldots, k_1} \) is the composition of successive symmetries with respect to circles \( a_{k_1} + r_{k_1} \mathbb{T}, a_{k_2} + r_{k_2} \mathbb{T}, \ldots, a_{k_p} + r_{k_p} \mathbb{T} \). In the sequence \( k_1, k_2, \ldots, k_p \) no two neighboring numbers are equal. The number \( p \) is called the level of the mapping. When \( p \) is even, these are Möbius transformations. If \( p \) is odd, we have anti-Möbius transformations, i.e. Möbius in \( \mathbb{T} \). These mappings \( \gamma_k \) are indexed as follows \( \gamma_0(z) = z, \gamma_1(z) = z^{*}_{11}, \gamma_2(z) = z^{*}_{22}, \ldots, \gamma_n(z) = z^{*}_{nn} \). Then \( \gamma_{n+1}(z) = z^{*}_{(21)}, \ldots \) and generate a Schottky group \( \mathcal{K} \). The next Theorem we state the existence and give a representation of the single-valued solution of the modified Dirichlet problem (3.2). It is based on [15, Thm. 4.12].

Theorem 3.3. Let \( m, n \in \mathbb{N} \setminus \{0\}, \alpha \in ]0, 1[, \) and let \( \phi \equiv (\phi_1, \ldots, \phi_n) \in C_{\ast}^{m,\alpha}(\mathbb{T}, \mathbb{C})^n \cap \mathbb{Z}^+ \). Let \( \mathbf{f} \equiv (f_1, \ldots, f_n) \in \prod_{j=1}^n C_{\ast}^{m,\alpha}(\phi_j(\mathbb{T}), \mathbb{R}) \). Let \( a_k \in \mathbb{C}, \) \( r_k \in \mathbb{R}^+, \) \( |a_k - a_j| > r_k + r_j, k \neq j, \) with \( k = 1, \ldots, n \), and let \( g \) be a conformal map of \( D_{\mathbf{a}, \mathbf{r}} \equiv \bigcap_{k=1}^n E[a_k + r_k \mathbb{T}] \) onto \( E[\phi] \). Let \( \alpha_j, j = 1, \ldots, n \) be the harmonic measures of \( D_{\mathbf{a}, \mathbf{r}} \). Let \( \lambda_k \in \mathbb{R} \) with \( k = 1, \ldots, n \). Let \( v \in E[\phi] \). Denote \( \omega \equiv g^{-1}(v) \in D_{\mathbf{a}, \mathbf{r}} \).

Then problem (3.2) associated to \( (\lambda_k)_{k=1}^n \in \mathbb{R}^n \) has a single-valued solution if and only if the following \( n - 1 \) linear independent equations are satisfied by \( (\lambda_k)_{k=1}^n \)

\[
\sum_{k=1}^n \int_{a_k + r_k \mathbb{T}} (f_k \circ g(\zeta) + \lambda_k) \frac{\partial \alpha_j}{\partial \nu}(\zeta)|d\zeta| = 0, \quad j = 1, 2, \ldots, n - 1,
\]

where \( \nu \) denotes the interior normal vector to the boundary of \( D_{\mathbf{a}, \mathbf{r}} \). In this case the solution is unique up to a purely imaginary constant and has the following
representation
\[
\tilde{F}(z) = \frac{1}{2\pi i} \sum_{k=1}^{n} \int_{a_k + r_k T} (f_k \circ g(\zeta) + \lambda_k) \left\{ \sum_{s>0} \left[ \frac{1}{\zeta - \gamma_s (\omega)} - \frac{1}{\zeta - \gamma_s (g^{-1}(z))} \right] \right\} + \\
(3.6) + \left( \frac{r_k}{\zeta - a_k} \right)^2 \sum_{s>0} \left[ \frac{1}{\zeta - \gamma_s (g^{-1}(z))} - \frac{1}{\zeta - \gamma_s (\omega)} \right] - \frac{1}{\zeta - g^{-1}(z)} d\zeta + \\
+ \frac{1}{2\pi} \sum_{k=1}^{n} \int_{a_k + r_k T} (f_k \circ g(\zeta) + \lambda_k) \frac{\partial A}{\partial \nu}(\zeta) |d\zeta| + i\zeta,
\]
where the function \( A \) can be chosen in the following way
\[
(3.7) A(\zeta) = \log |\zeta - u_{i(1)}^a - \alpha_1(\zeta) \log r_1 - \sum_{m=2}^{n} \alpha_m(\zeta) \log |u_{i(1)}^a - a_m|, \ \forall \zeta \in \text{cl} \mathbb{D}_{a,r}
\]
and where the summation in \( \sum_{s}^{'} \) (in \( \sum_{s}^{''} \)) is performed for all odd (respectively even) order elements \( \gamma_s \) of the corresponding Schottky group as above introduced.

The boundary values of a solution satisfy the following formulas.
\[
\tilde{F}(t_j) = f_j(t_j) + \lambda_j + \frac{p.v.}{2\pi i} \int_{\gamma_j^{-1}(t_j)} \frac{f_j \circ g(\zeta)}{\zeta - g^{-1}(t_j)} d\zeta - \frac{p.v.}{2\pi i} \int_{a_j + r_j T} \frac{f_j \circ g(\zeta)}{\zeta - g^{-1}(t_j)} d\zeta + \\
+ \frac{1}{2\pi} \sum_{k=1}^{n} \int_{a_k + r_k T} (f_k \circ g(\zeta) + \lambda_k) \left\{ \sum_{s>0} \left[ \frac{1}{\zeta - \gamma_s (\omega)} - \frac{1}{\zeta - \gamma_s (g^{-1}(t_j))} \right] \right\} + \\
(3.8) + \left( \frac{r_k}{\zeta - a_k} \right)^2 \sum_{s>0} \left[ \frac{1}{\zeta - \gamma_s (g^{-1}(t_j))} - \frac{1}{\zeta - \gamma_s (\omega)} \right] - \frac{1}{\zeta - g^{-1}(t_j)} d\zeta + \\
+ \frac{1}{2\pi} \sum_{k=1}^{n} \int_{a_k + r_k T} (f_k \circ g(\zeta) + \lambda_k) \frac{\partial A}{\partial \nu}(\zeta) |d\zeta| + i\zeta,
\]
for all \( t_j \in \phi_j(T) \) and \( j = 1, \ldots, n \), where \( \delta_{js} \) is the usual Kronecker symbol. In particular \( \tilde{F}_{|\phi_j} \in C_{*}^{m,\alpha}(\phi_j(T), \mathbb{C}) \) for all \( j \in \{ 1, \ldots, n \} \).

Then we have the following Theorem which summarizes the previous statements.

**Theorem 3.4.** Let \( m, n \in \mathbb{N} \setminus \{ 0 \} \), \( \alpha \in [0, 1] \). Let \( \phi \in C_{*}^{m,\alpha}(T, \mathbb{C}) \cap \mathbb{Z}^+ \) and let \( f \in \prod_{j=1}^{n} C_{*}^{m,\alpha}(\phi_j(T), \mathbb{R}), \ c \in \mathbb{R}^n \).

Then \( F \) is a (possibly multi-valued) solution of (3.1) for some \( \lambda \in \mathbb{R}^n \) if and only if \( F = F_0 + F + d, \) where \( d \in \mathbb{C}, \ F_0 \) is the multi-valued function introduced in (3.3) and \( F \) is a single-valued solution of problem (3.2) associated to \( f_j \equiv f_j - (\text{Re} F_0)_{|\phi_j(T)}, j = 1, \ldots, n, \) (which can be described by (3.6)). In this case \( F \) is single-valued if and only if \( c = 0 \). Moreover \( (\text{Re} F)_{|\phi_j(T)} \) and \( \tilde{F}_{|\phi_j(T)} \) belong to \( C_{*}^{m,\alpha}(\phi_j(T), \mathbb{C}) \) for all \( j = 1, \ldots, n \).
By the previous theorem it follows that the complex derivative of a solution $F$ of (3.1) is single-valued and does not depend on the particular solution we choose. By simply taking the tangential derivative of the boundary condition (a) of (3.1), we introduce a new boundary value problem for single-valued function $W$ and we prove that $F'$ is its unique solution.

**Proposition 3.5.** Let $m, n \in \mathbb{N} \setminus \{0\}$, $\alpha \in [0, 1]$. Let $\phi \in C^{m, \alpha}_r(T, \mathbb{C}) \cap \mathbb{Z}^+$ and let $f \in \prod_{j=1}^n C^{m, \alpha}_r(\phi_j(T), \mathbb{R})$, $c \in \mathbb{R}^n$. Let $F$ be a possibly multi-valued solution of the following boundary value problem

$$
\begin{align*}
F \in \mathcal{H} \left( \mathbb{E} \left[ \phi \right] \right) \cap C^1 \left( \mathbb{E} \left[ \phi \right] \right), \\
\lim_{z \to \infty} F'(z) = 0, \\
\langle \text{Re } F \rangle_{\phi_j(\gamma)} = f_j + \lambda_j, \\
\text{Im } \left\{ \int_{\phi_j} F'(\xi) d\xi \right\} = c_j, \quad j = 1, \ldots, n.
\end{align*}
$$

(3.9)

Then the complex derivative of $F$ is single-valued, does not depend on the particular solution of (3.9) we choose and satisfies the following boundary value problem

$$
\begin{align*}
W \in \mathcal{H} \left( \mathbb{E} \left[ \phi \right] \right) \cap C^0 \left( \mathbb{E} \left[ \phi \right] \right), \\
\lim_{z \to \infty} W(z) = 0, \\
\text{Re } \left\{ W(\phi_j(t)) \phi_j'(t) dt \right\} = \frac{d}{dt} \left\{ \int f_j(\phi_j(t)) dt \right\}, \\
\text{Im } \left\{ \int_{\phi_j} W(\xi) d\xi \right\} = c_j, \quad j = 1, \ldots, n.
\end{align*}
$$

(3.10)

Moreover problem (3.10) has a unique solution $W$, whose boundary values $W_{|_{\phi_j(\gamma)}}$ belong to $C^{m-1, \alpha}_r(T, \mathbb{C})$ for all $j = 1, \ldots, n$.

**Proof.** By Theorem 3.4 $F'$ is single-valued, does not depend on the particular solution of (3.9) we choose and satisfies conditions (a), (b) and (d) of (3.10). Now we prove that $F'$ satisfies also condition (c) of (3.10). Let $u := \text{Re } F$, $v := \text{Im } F$, $\phi_{j,1} := \text{Re } \phi_j$, $\phi_{j,2} := \text{Im } \phi_j$. By condition (c) of (3.9) and by Cauchy-Riemann equation it follows

$$
\frac{d}{dt} \left\{ u \left( \phi_{j,1}(e^{i\theta}), \phi_{j,2}(e^{i\theta}) \right) \right\} = \frac{d}{dt} \left\{ f_j \left( \phi_j(e^{i\theta}) \right) \right\},
$$

$$
\frac{\partial u}{\partial x} \left( \phi_{j,1}(e^{i\theta}) \right) \phi_{j,1}'(e^{i\theta}) e^{i\theta} - \frac{\partial v}{\partial x} \left( \phi_{j,1}(e^{i\theta}) \right) \phi_{j,2}'(e^{i\theta}) e^{i\theta} = \frac{d}{dt} \left\{ f_j \circ \phi_j \left( e^{i\theta} \right) \right\} e^{i\theta},
$$

which yields condition (c) of (3.10) after replacing $e^{i\theta}$ by $t$ (we have also used that $F' = \frac{du}{dt} + i\frac{dv}{dt}$).

By regularity part of Theorem 3.4, to conclude it suffices to prove the unique solvability of (3.10). Let $W$ be a solution of (3.10) with $u \equiv 0$ and $c = 0$. To see the unique solvability of (3.10) it suffices to show that $W \equiv 0$. We first prove that there exists $F \in \mathcal{H} \left( \mathbb{E} \left[ \phi \right] \right) \cap C^1 \left( \mathbb{E} \left[ \phi \right] \right)$ such that $F' = W$ in $\mathbb{E} \left[ \phi \right]$. By Lemma 2.5 it is enough to prove that the line integral $\int_{\phi_j} W(\xi) d\xi$ vanishes for all $j = 1, \ldots, n$.

By assumption $c = 0$. Hence $\text{Im } \left\{ \int_{\phi_j} W(\xi) d\xi \right\} = 0$. Moreover

$$
\text{Re } \int_{\phi_j} W(\xi) d\xi = \int_0^{2\pi} \text{Re } \left\{ W(\phi_j(e^{i\theta})) \phi_j'(e^{i\theta}) i e^{i\theta} \right\} d\theta
$$

$$
= \int_T \text{Re } \left\{ W(\phi_j(t)) \phi_j'(t) dt \right\} \left\{ \frac{d}{dt} \right\} = 0,
$$
because $f_j = 0$. Since $\lim_{z \to -\infty} F'(z) = 0$, the following expansion near $\infty$ holds

$$F'(z) = \sum_{s=1}^{\infty} \frac{a_{-s}}{z^s}.$$  

Since $\int_{\phi} F'(\xi) d\xi = 0$ for $j = 1, \ldots, n$, then $a_{-1} = 0$ and $\lim_{z \to -\infty} F(z) \in \mathbb{C}$.

We now prove that $F$ has constant real part on $\phi_j(\mathbb{T})$ for $j = 1, \ldots, n$. Indeed the following equations hold

$$\frac{d}{d\theta} \{\text{Re} F(\phi_j(e^{i\theta}))\} = \text{Re} \{W(\phi_j(e^{i\theta}))\phi_j'(e^{i\theta})ie^{i\theta}\} = 0, \quad \forall \theta \in [0, 2\pi],$$

by condition (c) of (3.10) and $f_j = 0$. Then $F$ solves the following Schwarz problem

$$\begin{align}
F & \in \mathcal{H}(\mathbb{E}[\phi]) \cap \mathcal{C}^0(\mathbb{E}[\phi]), \\
& \lim_{z \to -\infty} F(z) \in \mathbb{C}, \\
& \text{Re} F|_{s_j(T)} = \lambda_j, \quad \text{for some} \quad \lambda_j \in \mathbb{R} \quad \text{and} \quad j = 1, \ldots, n.
\end{align}$$

(3.11)

By Lemma 3.2 $F$ differs for a constant with respect to the zero solution corresponding to $(\lambda_j)|_{j=1}^n = 0$ in (3.11). Therefore $W(z) = F'(z) = 0$ for all $z \in \mathcal{C}[\phi]$.

4. Real Analyticity of an Operator Associated to the Schwarz Operator

In the previous section we have considered the following boundary value problem. Given $\phi \equiv (\phi_1, \ldots, \phi_n) \in C^{m,\alpha}(\mathbb{T}, \mathbb{C})^n \cap \mathcal{I}^+$, $f \equiv (f_1, \ldots, f_n) \in \prod_{j=1}^n C^{m,\alpha}(\phi_j(T), \mathbb{R})$, and $c \equiv (c_1, \ldots, c_n) \in \mathbb{R}^n$, we look for the single-valued holomorphic function $W \in \mathcal{C}^0(\mathbb{E}[\phi] \cup \{\infty\}, \mathbb{C})$ satisfying (3.10). By reducing this problem to a Schwarz problem and by the Mityushev & Rogosin result about solvability of the Schwarz problem for a multiply connected domain ([15]) we have showed that there exists a unique single-valued holomorphic function $W$ satisfying (3.10) and $W|_{s_j(T)} \in C^{m-1,\alpha}(\phi_j(T), \mathbb{C})$ for all $j = 1, \ldots, n$. Define now the operator

$$(4.1) \quad W[\phi, f, c] \equiv (W_1, \ldots, W_n) : [\phi, f, c] \mapsto \left( W|_{s_1(T)}, \ldots, W|_{s_n(T)} \right)$$

which assigns to each triple $[\phi, f, c]$ the boundary functions of the unique single-valued solution to (3.10).

We cannot study directly the regularity of the dependence of the operator $W[\phi, f, c]$ on $\phi, f$ and $c$ because the functional variables $f$ and $W[\phi, f, c]$ depend on $\phi$. As in [19] we introduce a new operator $W_*$ by replacing $W[\phi, f, c]$ and $f$ with their composition with $\phi$.

**Definition 4.1.** The modified operator $W_*$ from $(C^{m,\alpha}(\mathbb{T}, \mathbb{C})^n \cap \mathcal{I}^+) \times C^{m,\alpha}(\mathbb{T}, \mathbb{R})^n \times \mathbb{R}^n$ to $C^{m-1,\alpha}(\mathbb{T}, \mathbb{C})^n$ is defined by the equality

$$(4.2) \quad W_*[\phi, p, c] \equiv W[\phi, p \circ \phi^{-1}, c] \circ \phi.$$  

In order to study the regularity of $W_*$, we introduce a real analytic operator $A$ depending on $(\phi, p, c, G)$ and such that its zero level set is the graph of $W_*$. Then the result will follow by proving the well-posedness of the boundary value problem associated to the differential of $A$ with respect to $G$. 

Lemma 2.4 (iii) (satisfies $h$ phic in be the multi-valued function defined in (3.3). Then $1$

Let $G$ be the closed subspace of $C_*^{m-1,0}(T, C)$ and let $(\phi_0, p_0, c_0) \in A$. It suffices to prove that $W_*$ is real analytic in a neighbourhood of $(\phi_0, p_0, c_0)$ in $A$. Let $\Omega$ be a subset of $C_*^{m-1,0}(T, C)$ defined by

$$(3.3) \quad \Omega \equiv \left\{ u \in C_*^{m-1,0}(T, C)^n : u - P \left[ \phi_0, u \circ \phi_0^{-1} \right] \circ \phi_0 = 0 \right\}.$$  

By Lemmas 2.2 (iv), (v) and 2.4 (i), $\Omega$ is a closed subspace of $C_*^{m-1,0}(T, C)^n$. Let $\mathcal{V}$ be the closed subspace of $C_*^{m-1,0}(T, C)$ of all functions $q$ such that $\int_T(q(\zeta)/\zeta)d\zeta = 0$. Let $A$ be the operator of $A \times C_*^{m-1,0}(T, C)^n$ to $\Omega \times \mathcal{V}$ by the Implicit Function Theorem and by the Open Mapping Theorem, to conclude it suffices to prove that the operator $G \rightarrow A[\phi_0, \mathbf{0}, \mathbf{0}, G]$ is invertible from $C_*^{m-1,0}(T, C)^n$ to the target space of $A$. Let $d, e \in \mathbb{R}^n$ and let $(u, q, d + i e)$ be an arbitrary element of $\Omega \times \mathcal{V} \times C^n$. By Lemma 2.4 (iii) (3.3) be equal to $P[\phi_0, G \circ \phi_0^{-1}] \circ \phi_0$. Let $F_0$ be the multi-valued function defined in (3.3). Then $F_0$ is single-valued, holomorphic in $\mathbb{E}[\phi_0]$, null at infinity and, for a suitable choice of the coefficients $c_j$'s in (3.3), satisfies $\int_{\phi_0} F_0'(\xi)d\xi = d$. We set $h \equiv F_0' \circ \phi_0$ and $k(t) = e^{\{h(t) \phi_0'(t)it\} - 1/(2\pi)} \int_{\phi_0(t)} (Re \{h(\zeta) \phi_0'(\zeta)it\})/(i\zeta)d\zeta$ for all $t \in T$. By Lemmas 2.6, 2.2 (ii) and (iv), $h, k \in C_*^{m-1,0}(T, C)$. Since $k, q \in \mathcal{V}$ then there exist $\tilde{k}, \tilde{q} \in C_*^{m,0}(T, \mathbb{R})$ such that $\tilde{k}(t)it = h(t)it = q(t) = q(t)$ for all $t \in T$. Since $u \in \Omega$, Lemma 2.4 (ii) implies that $u_j \circ \phi_0^{-1}$ is the trace of a holomorphic map of $\mathbb{L}[\phi_0]$ for all $j = 1, \ldots, n$. In particular, $\int_{\phi_0}(u_j \circ \phi_0^{-1})(\xi)d\xi = 0$ and there exists $u_j \in C_*^{m,0}(\phi_0(T), C)$ such that $u_j = u_j \circ \phi_0^{-1}$ for all $j = 1, \ldots, n$. Simple computations show that the equality $A[\phi_0, \mathbf{0}, \mathbf{0}, G] = (u, q, d + i e)$ is equivalent to the system

$$(4.4) \quad \begin{cases} P \left[ \phi_0, (G - h - u) \circ \phi_0^{-1} \right] = 0, \\ Re \{G(t) - h(t) - u(t)\phi_0(t)it\} = \frac{d}{d\xi}\{\tilde{q}(t) - \tilde{k}(t) - Re w \circ \phi_0(t)\}it, t \in T, \\ Im \left\{ \int_{\phi_0}(G - h - u) \circ \phi_0^{-1}(\xi)d\xi \right\} = e. \end{cases}$$

By Proposition 3.5 system (4.4) has a unique solution $G \in C_*^{m-1,0}(T, C)^n$. \hfill \Box
Acknowledgment. The authors are indebted to the Italian ‘Istituto Nazionale di Alta Matematica’ ‘F.Severi’ and to the ‘Belarusian Fund for Fundamental Scientific Research’ for their financial support.

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